

Knowledge and Procedures

1. The following is an encrypted message: 8448856744756870

The following method was used to encrypt it:

- split the message into groups of four letters, then, for each group:
 - replace each letter with the number corresponding to its position in the alphabet to get a, b, c, d
 - arrange these numbers into a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
 - find the matrix $\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix}$
 - arrange the elements of $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ as a sequence e, f, g, h
- reassemble the message in order

Decipher the message.

2. Find the volume of a parallelepiped if one corner is at $(0, 2, 1)$ and edges run from that corner to $(0, 3, -1)$, $(2, 3, 0)$ and $(1, 4, 3)$.
3. Use finite differences to find a formula for $\sum_{r=1}^x r^2$.

Modelling and Problem Solving

4. Find a matrix which will produce a single transformation of the x - y plane equivalent to the following sequence:
- a rotation of 70° about the origin
 - a reflection in the x -axis
 - a rotation of 130° about the origin
 - a reflection in the y -axis
 - a rotation of 60° about the origin
 - a dilation by a factor of 2 from the origin
5. If the sum of the sequence $1, 3, 7, 15, 31, 63 \dots$ to a million terms is $2^x + y$, find a value of $x + y$.
6. Poggles is in the northern hemisphere 10 000 km by air from Brisbane (27°S 153°E) and 10 000 km by air from Singapore (1°N 104°E). Find his latitude and longitude. [Note that 10 000 km is one quarter of the circumference of the Earth.]

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$$\textcircled{1} \tilde{A} = \begin{pmatrix} 84 & 48 \\ 85 & 67 \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} 44 & 75 \\ 68 & 70 \end{pmatrix}$$

$$\tilde{A} - \begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix} = \begin{pmatrix} 34 & -2 \\ 35 & 17 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} &= \frac{1}{2+1} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 34 & -2 \\ 35 & 17 \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 69 & 15 \\ 36 & 36 \end{pmatrix} \\ &= \begin{pmatrix} 23 & 5 \\ 12 & 12 \end{pmatrix} \end{aligned}$$

This is WELL

$$\tilde{B} - \begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix} = \begin{pmatrix} -6 & 25 \\ 18 & 20 \end{pmatrix}$$

$$\begin{aligned} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -6 & 25 \\ 18 & 20 \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 12 & 45 \\ 42 & 15 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 15 \\ 14 & 5 \end{pmatrix} \end{aligned}$$

This is DONE

So the message is WELL DONE

(2) The edge vectors are

$$(0, 3, -1) - (0, 2, 1) = (0, 1, -2)$$

$$(2, 3, 0) - (0, 2, 1) = (2, 1, -1)$$

$$(1, 4, 3) - (0, 2, 1) = (1, 2, 2)$$

The volume is the scalar triple product, which is

$$\begin{vmatrix} 0 & 1 & -2 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= -1(2 \times 2 + 1 \times 1) - 2(2 \times 2 - 1 \times 1)$$

$$= -5 - 6$$

$$= -11$$

\therefore The volume is 11

$$(3) \quad y = \sum_{r=1}^x r^2$$

x	0	1	2	3	4	5	6
y	0	1	5	14	30	55	91

$$\begin{array}{cccccc} 1 & 4 & 9 & 16 & 25 & 36 \\ 3 & 5 & 7 & 9 & 11 & \\ & 2 & 2 & 2 & & \end{array}$$

Δ^3 is constant, so it is a cubic of the form

$$y = ax^3 + bx^2 + cx + d$$

$$\Delta^3 = 6a$$

$$2 = 6a$$

$$a = \frac{1}{3}$$

When $x=0, y=0 \therefore d=0$

So the formula is $y = \frac{1}{3}x^3 + bx^2 + cx$

When $x=1, y=1$

$$\therefore 1 = \frac{1}{3} + b + c$$

$$b + c = \frac{2}{3} \quad \dots \textcircled{1}$$

When $x = 2$ $y = 5$

$$\therefore 5 = \frac{1}{3} \times 8 + 4b + 2c$$

$$4b + 2c = 2\frac{2}{3} \dots \textcircled{2}$$

$$2b + 2c = 1\frac{1}{3} \dots \textcircled{3}$$

$$\textcircled{1} \times 2 \Rightarrow$$

$$\textcircled{2} - \textcircled{3} \Rightarrow$$

$$2b = 1$$

$$b = \frac{1}{2} \dots \textcircled{4}$$

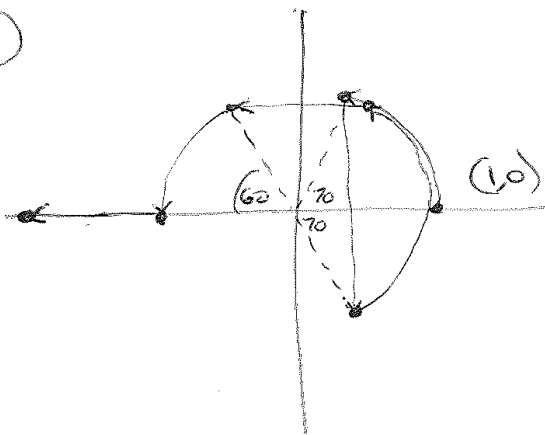
$$\textcircled{4} \text{ into } \textcircled{2} \Rightarrow$$

$$\frac{1}{2} + c = 2\frac{2}{3}$$

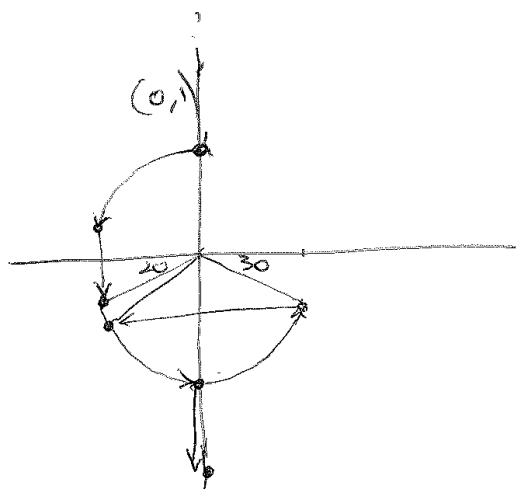
$$c = \frac{1}{6}$$

$$\therefore \text{The formula is } y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x$$

(4)



$(1, 0)$ is transformed to $(-2, 0)$



$(0, 1)$ is transformed to $(0, -2)$

So the transformation matrix is $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$$\textcircled{5} \quad 1 + 3 + 7 + 15 + 31 + 63 + \dots$$

$$= 2^1 - 1 + 2^2 - 1 + 2^3 - 1 + 2^4 - 1 + \dots + 2^{1000000} - 1$$

$$= 2^1 + 2^2 + 2^3 + \dots + 2^{1000000} - 1000000$$

$$= 2^{1000001} - 2 - 1000000$$

$$= 2^{1000001} - 1000002$$

$$\text{So } x = 1000001, \quad y = -1000002$$

$$\therefore x + y = -1$$

- (6) Let the vector from the centre of the Earth to Poggie be \underline{p}
 Brobarnie be \underline{b}
 Singapore be \underline{s}

$$\underline{p} = \underline{s} \times \underline{b}$$

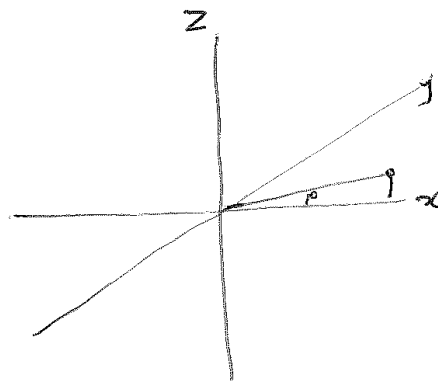
Let the radius of the Earth be 1
 Let the zero of longitude pass through Singapore

Then, for \underline{s}

$$\theta = 0$$

$$\phi = 1^\circ$$

$$r = 1$$



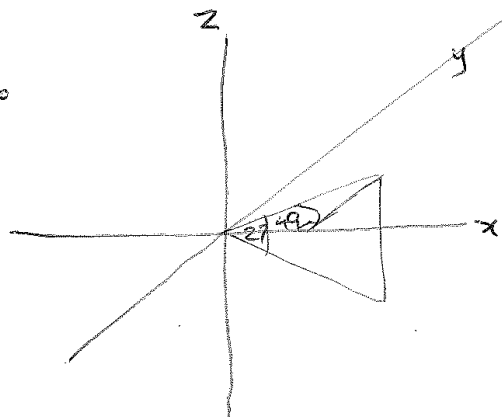
$$\begin{aligned} \text{So } \underline{s} &= \cos 1^\circ \underline{i} + \sin 1^\circ \underline{k} \\ &= 0.9998 \underline{i} + 0.0175 \underline{k} \end{aligned}$$

For \underline{b}

$$\theta = 153 - 104 = 49^\circ$$

$$\phi = -27^\circ$$

$$r = 1$$



$$\begin{aligned} \text{So } \underline{b} &= \cos 27^\circ \cos 49^\circ \underline{i} \\ &\quad + \cos 27^\circ \sin 49^\circ \underline{j} \\ &\quad - \sin 27^\circ \underline{k} \\ &= 0.5846 \underline{i} + 0.6725 \underline{j} - 0.4540 \underline{k} \end{aligned}$$

$$\underline{r} = \underline{s} \times \underline{b}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0.9998 & 0 & 0.0175 \\ 0.5846 & 0.625 & 0.4540 \end{vmatrix}$$

$$= -0.0118 \underline{i} + 0.4657 \underline{j} + 0.6724 \underline{k}$$

Peggle's position

$$\phi = \sin^{-1} 0.6724 = 42.4^\circ \text{ N}$$

$$\theta = \tan^{-1} \frac{0.0118}{0.4657} + 90^\circ$$

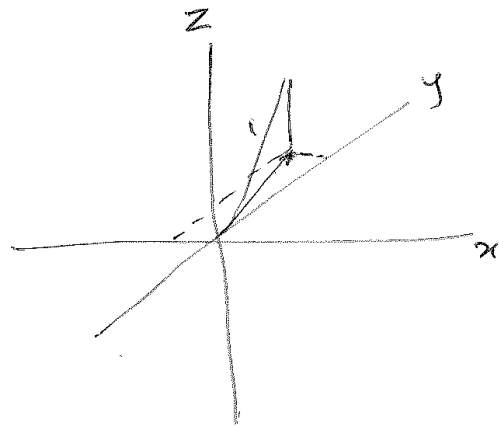
$$= 91.45^\circ$$

\therefore latitude is 42° N

longitude is $104 + 91^\circ \text{ E}$

$$= 195^\circ \text{ E}$$

$$= 165^\circ \text{ W}$$



So Peggles is at $42^\circ \text{ N } 165^\circ \text{ W}$