Vectors and Projectiles

V1. The position of a particle is given by the vector $12t\boldsymbol{i} - (t^3 - 8)\boldsymbol{j}$, where t is the time in seconds and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors towards the east and north respectively.

a. Find expressions for the velocity and acceleration of the particle
b. Find the position, velocity and acceleration of the particle when $t = 3$
c. Use your calculator to plot the path of the particle
d. Find the equation of the path in the form $y = f(x)$
e. In what direction is the particle moving when it is due east of its starting point?
f. Find the acceleration when it is moving in the direction 60° south of east.

V2. The velocity of a seagull is given by the vector $(8 - 4t)\boldsymbol{i} + (t - 1)^2\boldsymbol{j}$, where t is the time in seconds and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors to the right and upwards respectively. The seagull is at $10\boldsymbol{i} + 2\boldsymbol{j}$ when $t = 0$.

a. Find the expression for its position at any time t
b. Plot its path for the first 4 s
c. Find its angle of climb when $t = 4$
d. Find its position, velocity and acceleration when it is moving vertically upwards

V3. The acceleration of a cannonball is $-10\boldsymbol{j}$. Its initial position and velocity are $-10\boldsymbol{i} + 2\boldsymbol{j}$ and $10\boldsymbol{i} + 30\boldsymbol{j}$ respectively. $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors to the right and up respectively.

a. Find expressions for its velocity and position at time t
b. Find its position when it is moving horizontally
c. Find its position and velocity when it is 12 units lower than its starting height

V4. A cannon ball is fired out to sea from the top of a cliff 100 m above sea level. It is fired at 60 m/s 30° above the horizontal. Find vector expressions for the acceleration, velocity and position of the cannonball t seconds after firing. Hence find how far horizontally from the cannon the cannonball hits the sea. Assume the cannon to be at the origin and g to be 9.8 m/s^2. Also assume that there is no air resistance.

V5. A cannon is being fired to hit a target 300 m in front of it and 60 m higher than it. The cannon is pointed 45° from the horizontal. Use vector calculus to find the speed at which the cannonball should be fired?

Answers

V1. a. $\boldsymbol{v} = 12\boldsymbol{i} - 3t^2\boldsymbol{j}$, $\boldsymbol{a} = -6t\boldsymbol{j}$ b. $\boldsymbol{p} = 36\boldsymbol{i} - 19\boldsymbol{j}$, $\boldsymbol{v} = 12\boldsymbol{i} - 27\boldsymbol{j}$, $\boldsymbol{a} = -18\boldsymbol{j}$
d. $y = 8 - x^3/1728$ e. SE f. $-15.79\boldsymbol{j}$

V2. a. $(8t - 2t^2 + 10)i + (\frac{1}{3}t^3 - t^2 + t + 2)j$ c. 48.4° d. $\boldsymbol{p} = 18\boldsymbol{i} + \frac{8}{3}\boldsymbol{j}$, $\boldsymbol{v} = \boldsymbol{j}$, $\boldsymbol{a} = -4\boldsymbol{i} + 2\boldsymbol{j}$

V3. a. $\boldsymbol{v} = 10\boldsymbol{i} + (30 - 10t)\boldsymbol{j}$, $\boldsymbol{p} = (10t - 10)\boldsymbol{i} + (-5t^2 + 30t + 2)\boldsymbol{j}$ b. $20\boldsymbol{i} + 47\boldsymbol{j}$ c. $\boldsymbol{p} = 53.76\boldsymbol{i} - 10\boldsymbol{j}$,
$\boldsymbol{v} = 10\boldsymbol{i} - 33.76\boldsymbol{j}$

V4. $\boldsymbol{a} = -9.8\boldsymbol{j}$, $\boldsymbol{v} = 51.96\boldsymbol{i} + (30 - 9.8t)\boldsymbol{j}$, $\boldsymbol{p} = 51.96t + (30t - 4.9t^2)j$, 442 m

V5. 60.6 m/s

Solutions

Vectors and Projectiles

Q1 Let the position of the particle be $\underset{\sim}{r}(t)$

$\underset{\sim}{r} = 12t\underset{\sim}{i} - (t^3-8)\underset{\sim}{j}$

a) velocity, $v = \frac{dr}{dt} = 12i - (3t^2)\underset{\sim}{j}$

acceleration, $a = \frac{dv}{dt} = -6t\underset{\sim}{j}$

b) When $t=3$ $\quad \underset{\sim}{r} = 12\times3\underset{\sim}{i} - (3^3-8)\underset{\sim}{j}$

$= 36\underset{\sim}{i} - 19\underset{\sim}{j}$

$v = 12i - 3\times3^2\underset{\sim}{j}$

$= 12i - 27\underset{\sim}{j}$

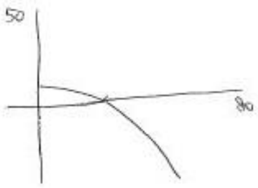
$a = -6\times3\underset{\sim}{j}$

$= -18\underset{\sim}{j}$

c) Using a parametric function with

$X_{t1} = 12t \quad$ and $\quad Y_{t1} = -t^3+8$

we get



d) Let the y component at any time be y

Let the x component be x

$y = -(t^3-8) \quad x = 12t$

$t = \frac{x}{12}$

$\therefore y = -\left(\frac{x}{12}\right)^3 + 8$

$y = -\frac{x^3}{1728} + 8$

e) When the particle is due east of its starting point,

$y = 0$

$\therefore t^3 - 8 = 0$

$t^3 = 8$

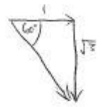
$t = 2$

Then $v = 12i - 3\times4j$

$= 12i - 12j$

Thus it is moving exactly SE

f) When it is moving $60°$ South of east, the y-component of velocity is $-\sqrt{3}$ times the x-component



ie. $3t^2 = \sqrt{3}\times12$

$t^2 = \frac{12}{\sqrt{3}} = 4\sqrt{3} = 6.928\ s^2$

$t = 2.632\ s$

Then the acceleration is $-6\times2.632\ \underset{\sim}{j}$

$= -15.79\underset{\sim}{j}$

Q2 (a) velocity, $v = (8-4t)\underset{\sim}{i} + (t-1)^2\underset{\sim}{j}$

position, $\underset{\sim}{r} = \int (8-4t)\underset{\sim}{i} + (t^2-2t+1)\underset{\sim}{j}\, dt$

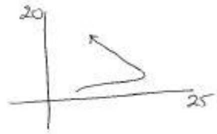
$= (8t - 2t^2 + c)\underset{\sim}{i} + \left(\frac{t^3}{3} - t^2 + t + d\right)\underset{\sim}{j}$

When $t=0$ $\quad \underset{\sim}{r} = 10i + 2j$

$\therefore c = 10$ and $d = 2$

So $\underset{\sim}{r} = (8t - 2t^2 + 10)\underset{\sim}{i} + \left(\frac{t^3}{3} - t^2 + t + 2\right)\underset{\sim}{j}$

(b) Graphing parametrically gives



(c) When $t=4$ $\quad \underset{\sim}{v} = -8i + 9\underset{\sim}{j}$

Angle of climb $= \tan^{-1} 9/8$

$= 48.4°$

(d) It is moving vertically upwards when $8-4t=0$

ie when $t=2$

Then $\underset{\sim}{r} = (16-8+10)\underset{\sim}{i} + \left(\frac{8}{3} - 4 + 2 + 2\right)\underset{\sim}{j}$

$\underset{\sim}{r} = 18\underset{\sim}{i} + 8/3\underset{\sim}{j}$

$\underset{\sim}{v} = (2-1)^2\underset{\sim}{j}$

$= \underset{\sim}{j}$

acceleration $= -4i + (2t-2)\underset{\sim}{j}$

$= -4i + 2\underset{\sim}{j}$

V3(a) acceleration $a = -10j$

velocity, $v = ci + (-10t + d)j$

When $t = 0$ $v = 10i + 30j$

$\therefore c = 10,\ d = 30$

$\therefore v = 10i + (-10t + 30)j$

position, $r = (10t + e)i + (-5t^2 + 30t + f)j$

When $t = 0$ $r = -10i + 2j$

$\therefore e = -10,\ f = 2$

$\therefore r = (10t - 10)i + (-5t^2 + 30t + 2)j$

(b) It is moving horizontally when $-10t + 30 = 0$

ie when $t = 3$

Its position then is $20i + 47j$

(c) It is 12 units below its starting position when $(-5t^2 + 30t + 2)j = -10j$

$$-5t^2 + 30t + 2 = -10$$

$$-5t^2 + 30t + 12 = 0$$

$$5t^2 - 30t - 12 = 0$$

$$t = \frac{30 \pm \sqrt{900 + 240}}{10}$$

$$= 3 \pm \sqrt{11.4}$$

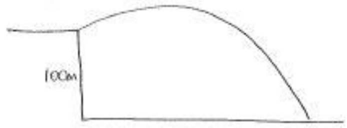
$$= 3 \pm 3.376$$

Take the positive solution $t = 6.376$

Then $r = 53.76i - 10j$

$v = 10i - 33.76j$

V4



Let i be the unit vector to the right
j be the unit vector upward

acceleration, $a = -9.8j$

velocity, $v = ci + (-9.8t + d)j$

When $t = 0$ $v = 60\cos 30 i + 60 \sin 30 j$

$= 51.96i + 30j$

$\therefore c = 51.96 \quad d = 30$

$\therefore v = 51.96i + (-9.8t + 30)j$

position $p = (51.96t + e)i + (-4.9t^2 + 30t + f)j$

When $t = 0$ $p = 0i + 0j$

$\therefore e = f = 0$

$\therefore p = 51.96t\,i + (-4.9t^2 + 30t)j$

It hits the sea when $(-4.9t^2 + 30t) = -100$

ie when $-4.9t^2 + 30t + 100 = 0$

ie when $4.9t^2 - 30t - 100 = 0$

ie when $t = \dfrac{30 \pm \sqrt{900 + 1920}}{9.8}$

$$t = \frac{30 \pm \sqrt{2820}}{9.8}$$

$$t = 8.48s \text{ or } -2.358s$$

Taking the positive solution, $t = 8.48s$

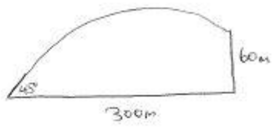
At this time, the horizontal position is

51.96×8.48

$= 442m$

$\therefore$ It enters the sea 442m horizontally from the cannon.

V5



Let the speed at which it is fired be v.

~~acceleration $\underset{\sim}{a} = -9.8\underset{\sim}{j}$~~

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors right and up respectively

acceleration, $\underset{\sim}{a} = -9.8\underset{\sim}{j}$

velocity, $\underset{\sim}{v} = c\underset{\sim}{i} + (-9.8t + d)\underset{\sim}{j}$

When $t = 0$ $\underset{\sim}{v} = \frac{1}{\sqrt{2}}v\,\underset{\sim}{i} + \frac{1}{\sqrt{2}}v\,\underset{\sim}{j}$

$\therefore c = \frac{1}{\sqrt{2}}v, \quad d = \frac{1}{\sqrt{2}}v$

$\therefore \underset{\sim}{v} = \frac{1}{\sqrt{2}}v\,\underset{\sim}{i} + \left(-9.8t + \frac{1}{\sqrt{2}}v\right)\underset{\sim}{j}$

~~position~~ Taking the cannon as the origin

$\underset{\sim}{r} = \left(\frac{1}{\sqrt{2}}vt + e\right)\underset{\sim}{i} + \left(-4.9t^2 + \frac{1}{\sqrt{2}}vt + f\right)\underset{\sim}{j}$

When $t = 0$ $r = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

$\therefore e = f = 0$

$\therefore \underset{\sim}{r} = \frac{1}{\sqrt{2}}vt\,\underset{\sim}{i} + \left(-4.9t^2 + \frac{1}{\sqrt{2}}vt\right)\underset{\sim}{j}$

When the horizontal displacement is 300

$\frac{1}{\sqrt{2}}vt = 300$

$t = \frac{300\sqrt{2}}{v}$

Then the vertical displacement is

$-4.9 \times \left(\frac{300\sqrt{2}}{v}\right)^2 + \frac{1}{\sqrt{2}}v \times \frac{300}{\frac{1}{\sqrt{2}}v} = 60$

$-\frac{4.9 \times 90\,000 \times 2}{v^2} + 300 = 60$

$-\frac{4.9 \times 90\,000 \times 2}{v^2} = -240$

$4.9 \times 90\,000 \times 2 = 240 \quad v^2$

$v^2 = \frac{4.9 \times 90\,000 \times 2}{240}$

$= 3675$

$v = 60.62$ m/s