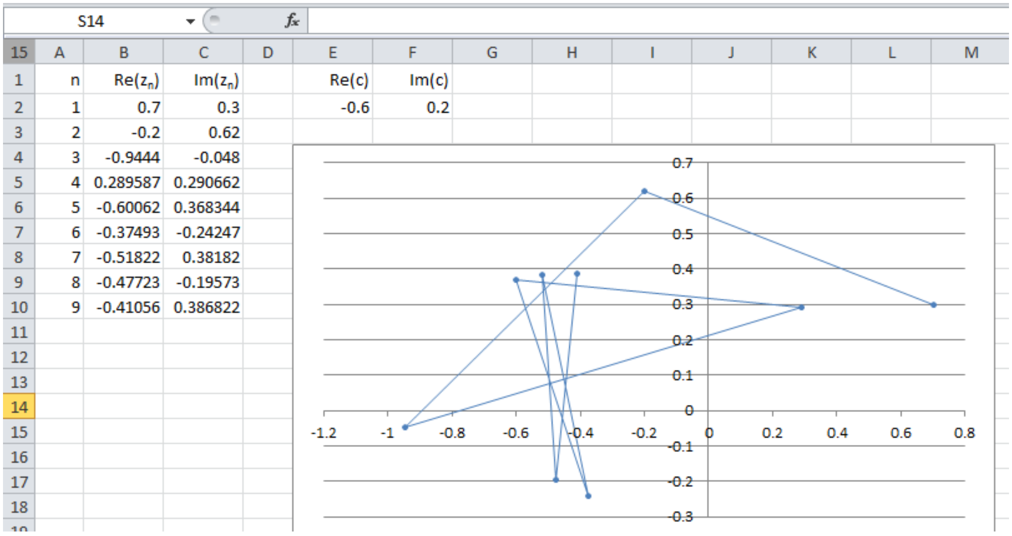
Knowledge and Procedures

Q1. Draw up by hand an Argand diagram with real and imaginary axes from –1 to 1.
Plot the point $z_1 = 0.8 + 0.1i$. Let $c = -0.6 + 0.3i$.

Calculate $z_2 = z_1^2 + c$ by hand (show working) and plot z_2 on the diagram.
Join z_1 to z_2 with a line.
Calculate $z_3 = z_2^2 + c$ with a calculator and plot z_3. Join z_2 to z_3 with a line.
Calculate $z_4 = z_3^2 + c$ and plot z_4. Join z_3 to z_4 with a line.
Calculate $z_5 = z_4^2 + c$ and plot z_5. Join z_4 to z_5 with a line.

We can continue this process indefinitely with $z_n = (z_{n-1})^2 + c$ and the resulting sequence of numbers produces a trajectory around the complex plane. This job can be tedious by hand, however, so we will use a spreadsheet.

Q2. Produce a spreadsheet in the format shown below to do the same job. (Again use $z_1 = 0.8 + 0.1i$, $c = -0.6 + 0.3i$; the one below is for $z_1 = 0.7 + 0.3i$ and $c = -0.6 + 0.2i$.)
n should range from 1 to 40 rather than 1 to 9.



	A	B	C	D	E	F
1	n	Re(z_n)	Im(z_n)		Re(c)	Im(c)
2	1	0.7	0.3		-0.6	0.2
3	2	-0.2	0.62			
4	3	-0.9444	-0.048			
5	4	0.289587	0.290662			
6	5	-0.60062	0.368344			
7	6	-0.37493	-0.24247			
8	7	-0.51822	0.38182			
9	8	-0.47723	-0.19573			
10	9	-0.41056	0.386822			

a) Write the formulas used in cells B3 and C3.

b) The graph should show the trajectory of z on the complex plane. To generate the graph, plot the real and imaginary parts as a scatter plot with straight lines. Ensure that the lines are at the thinnest width possible to help see the result. Print out the spread sheet and the graph for inclusion with your work.

c) Describe the trajectory. You may wish to extend the values of n further to check the validity of your description.

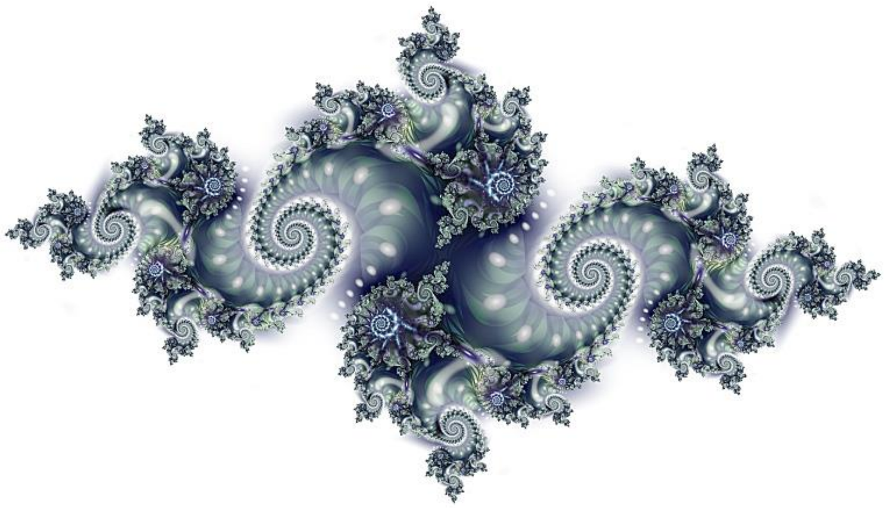
Q3. Replace z_1 with $1 + 0.1i$, and print out the sheet again (with the graph). Choose a suitable view window to effectively show what is happening.
Compare the two trajectories with particular reference to escape and prisoner sets.
(You may need to research the meaning of these words.)

Modelling and Problem Solving

Q4. You have done some of the calculations for producing a Julia set – a type of mathematically generated fractal. Research Julia sets and write a mathematical article of around 300 words on them. Your article should discuss:

- what they are and how they are produced
- how you would continue the work in Q1-3 to produce one
- interesting features of their shape and symmetry
- how the value of c affects the shape

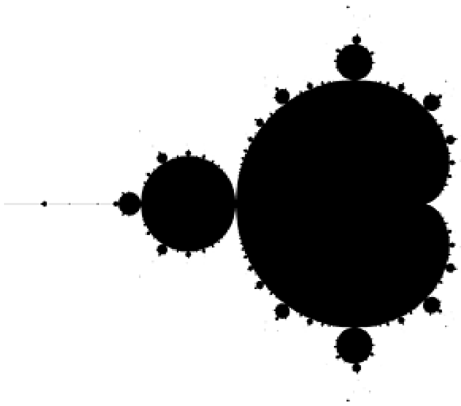
Include pictures in your article.



Q5. Write a mathematical article of around 300 words on the Mandelbrot set. Your article should discuss:

- what it is and how it is produced
- interesting features of its shape and symmetry
- how it relates to Julia sets

Include pictures.



Q6. Briefly discuss the relationship between fractals, chaos theory and the butterfly effect. You will probably need to do some research into this first. Again, you should write about 300 words, but feel free to write more if you have a lot to say.

Solutions

Q1. $z_2 = 0.03 + 0.46i$
$z_3 = -0.81 + 0.33i$
$z_4 = -0.05 - 0.23i$
$z_5 = -0.65 + 0.32i$

Q2. a) B3: =B2^2-C2^2+$E$2
C3: =2*B2*C2+$F$2

b) Spread sheet

c) The trajectory spirals in towards a point near $-0.43 + 0.16i$

Q3. In this trajectory, the point moves around seemingly randomly, but generally getting further from the origin. In Q1 z_1 is part of the prisoner set; in Q2, z_1 is part of the escape set.

Q4. Suppose we take a fixed complex number c and another complex number z_1. Then we calculate $z_2 = z_1^2 + c$, $z_3 = z_2^2 + c$ and so on. If z stays permanently near the origin, z_1 is said to be a prisoner; otherwise it is said to be an escapee. The Julia set for that value of c is the set of all prisoners, i.e. the prisoner set. These numbers form a shape on the complex plane.

Usually, the prisoner set is coloured black and the escape set is coloured according to how many iterations it takes for z to have a modulus greater than 2.

To continue from Q1-Q3 to make the Julia set for $c = 0.6+0.3i$, we would have to test all values of z within 2 of the origin to see if they are part of the prisoner set. We would do this by iterating say 200 times or until they escaped. As there are an infinite number of values of z, we might test on a 0.001 by 0.001 grid.

Julia sets have 2-fold rotational symmetry about the origin. Real values for c produce mirror symmetry about the axes as well.

The most interesting feature of their shape is that, for some values of c, their edge is very convoluted or even infinitely convoluted forming a fractal with infinite perimeter. The convolutions tend to repeat approximately at all scales as one zooms into them.

The shape of the Julia set varies with the value of c. For c close to the origin, the Julia set is a single region with a fairly simple boundary. For values of c distant from the origin, the Julia set consists of an infinite number of infinitely small isolated regions called 'dust'. The former type of Julia set is said to be connected, the latter disconnected. For intermediate values of c, the boundary tends to be very convoluted.

Pictures

Q5. The Mandelbrot Set consists of all the values of c for which the Julia set is connected. If a Julia set is connected, $z_1=0$ will always be in the prisoner set; if the Julia set is disconnected, $z_1=0$ will always be in the escape set. So the Mandelbrot set is all the values of c for which $z_1=0$ remains bounded under the iteration $z_n = z_{n-1}^2+c$.

The Mandelbrot set has a shape reminiscent of circles with smaller circles attached and smaller circles attached to those and so on. The edge is infinitely convoluted making it a fractal. Zooming in reveals repetitions of similar shapes including approximate smaller versions of the entire Mandelbrot set.

The set has bilateral symmetry about the real axis.

Pictures.

Q6. In a fractal like a convoluted Julia set, a very small change in z_1 can make a very large difference in subsequent values of z. A very small change can cause the point to move between the prisoner set and the escape set a very large number of times.

This makes subsequent behaviour seem chaotic, even though it is precisely determined. A very small measurement error in the position of z_1 can change subsequent z values totally.

Chaos theory is the study of systems in which differences in initial conditions of less than the sensitivity of measurements used to determine it can totally change subsequent states of the system. An example of such a system is the state of the atmosphere. A tiny change in initial conditions, for instance that produced by a butterfly flapping its wings in Japan, can make the difference between Florida having a hurricane and Florida having a sunny day a few months down the track. This is called the butterfly effect.

Marking Notes

Q1

Response must include

- correct argand diagram with axes marked
- calculation of z_1, z_2, and z_3
- a line drawn showing the connection of z_1, z_2, and z_3

Q2

Response must include

- development of Excel formulas through generalisation of formula and formulas written
- spreadsheet accurately developed with values for z_1 and c from question 1 with accompanying graph

Q3

Response must include

- graph and data for different value of z_1
- comparison of the two trajectories with mention of escape and prisoner sets

Q4

Response must include

- details on what Julia sets are and how they are produced
- how the first three questions help develop a Julia set
- interesting features of the shape and symmetry of Julia Sets
- the affect of the c value on the shape
- approx 300 words

Q5

Response must include

- what a Mandelbrot set is and how its produced
- interesting features of the shape and symmetry of Mandelbrot sets
- how they relate to Julia sets
- approx 300 words

Q6

Response must include

- approx 300 words
- the relationship that exists between fractals, chaos theory and the butterfly effect
- description of each of fractals, chaos theory and the butterfly effect