Question 1

Write T (true) or F (false) for each of the following:

(a) $-6 \in W$ (b) $\sqrt{-3} \in R$ (c) i^6 is an imaginary number

(d) $\frac{1}{6.7}$ is rational (e) $3 + 2\sqrt{3}$ is a complex number

Question 2

Define 'rational number'

Question 3

Prove that $\sqrt{5}$ is irrational. [You may assume that if x^2 is a multiple of n, then x is a multiple of n.]

Question 4

Express in simplest form:

(a) $\sqrt{45}$ (b) $\frac{\sqrt{48}}{\sqrt{75}}$ (c) $\frac{3\sqrt{5}}{2\sqrt{5}+1}$

Question 5

(a) Give $(1 - i) + (3 - 2i)$ in Cartesian form. Show working.

(b) Give $(3 - 2i) \times (1 + i)$ in Cartesian form. Show working.

Question 6

(a) Convert $3 - 3i$ to polar form. Show working.

(b) Convert $2 \text{ cis } \pi/6$ to Cartesian form. Show working.

Question 7

(a) Write $(\text{cis } \pi/3)^{20} \div (2 \text{ cis } 3\pi/4)$ in polar form. Show working.

(b) Write $\sqrt{4 \text{ cis } \pi}$ in Cartesian form. Show working.

Question 8

If $A = \begin{pmatrix} 4 & -2 \\ a & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}$,

find (a) $3A$ (b) $A + B$ (c) AB (d) A' (e) $|A|$ (f) A^{-1}

END OF PAPER ☺

Question 1

Use an algebraic method to find $\sqrt{i}$ in Cartesian form.

Question 2

Prove that $\overline{wz} = \overline{w}\,\overline{z}$ where $w, z \in \mathbb{C}$

Question 3

Solve for $\boldsymbol{X}$: $\begin{pmatrix} 2 & 3 & 0 \\ -1 & 5 & 1 \end{pmatrix} + 2\boldsymbol{X} = 3\begin{pmatrix} 0 & 1 & 0 \\ 4 & 2 & 5 \end{pmatrix}$

Question 4

If $\boldsymbol{A} = \begin{pmatrix} 4 & -2 \\ a & 1 \end{pmatrix}$, for what real values of a does $\boldsymbol{A}^{-1}$ exist?

Question 5

If $\boldsymbol{A} = \begin{pmatrix} 5 & a \\ -10 & -4 \end{pmatrix}$ and $\boldsymbol{A}^2 = \boldsymbol{A}$, find a.

Question 6

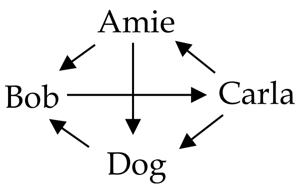
Use matrix methods as far as possible to solve the following problem.
Sawston High School bought 80 maths books, 30 physics books and 45 chemistry books for \$8505. Drillham High School bought 50 maths books, 10 physics books and 20 chemistry books for \$4190. Hammersmith High School bought 66 maths books, 27 physics books and 22 chemistry books for \$6256. Use matrix methods to find what Spannerville High School would pay for 56 maths books, 29 physics books and 19 chemistry books.

You may do the matrix operations on a calculator, but show all other working.

What assumptions do you make in reaching your answer?

Question 7

A round robin contest between four karate students produced the following digraph



where A→ B means A beat B

Use dominance matrices to rank the players, justifying the decisions you make.
What are the strengths and limitations of the method you used?

Question 8

Express with a rational denominator: $\dfrac{1}{2\sqrt{5} + \sqrt{15} + 2\sqrt{3} + 3}$

END OF PAPER ☺

Q1 a) F
b) F
c) F
d) T
e) T

Q2 A rational number is a number which can be written in the form a/b where $a, b \in \mathbb{Z}$

Q3 Assume that $\sqrt{5}$ is rational.

Then $\sqrt{5}$ can be written as a/b where $a, b \in \mathbb{Z}$

and a and b have no common factors.

Thus $\frac{a}{b} = \sqrt{5}$

$\frac{a^2}{b^2} = 5$

$a^2 = 5b^2$

$\therefore$ a^2 has a factor of 5

$\therefore$ a has a factor of 5

$\therefore$ a can be written as $5k$

Then $(5k)^2 = 5b^2$

$25k^2 = 5b^2$

$5k^2 = b^2$

$\therefore$ b^2 has a factor of 5

$\therefore$ b has a factor of 5

$\therefore$ a and b have a common factor

This contradicts the original assumption

$\therefore$ $\sqrt{5}$ is not rational

Q4

a) $\sqrt{45} = \sqrt{5}\sqrt{9} = 3\sqrt{5}$

b) $\dfrac{\sqrt{48}}{\sqrt{75}} = \dfrac{\sqrt{16}\sqrt{3}}{\sqrt{25}\sqrt{3}} = \dfrac{4}{5}$

c) $\dfrac{3\sqrt{5}}{2\sqrt{5}+1} = \dfrac{3\sqrt{5}}{2\sqrt{5}+1} \times \dfrac{2\sqrt{5}-1}{2\sqrt{5}-1}$

$= \dfrac{6\times5-3\sqrt{5}}{20-1}$

$= \dfrac{30-3\sqrt{5}}{19}$

~~d)~~

Q5

a) $(1-i)+(3-2i)$

$= 4-3i$

b) $(3-2i)\times(1+i)$

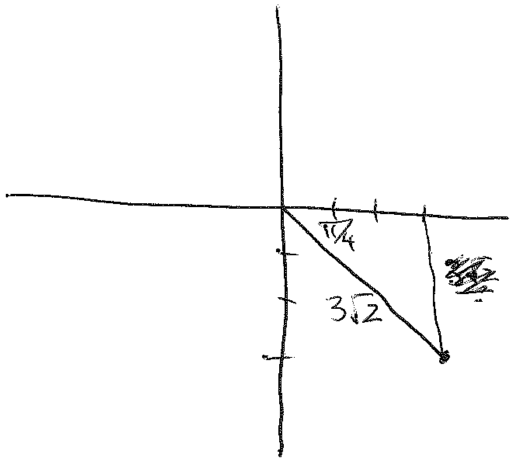
$= 3+3i-2i-2i^2$

$= 3+i+2$

$= 5+i$

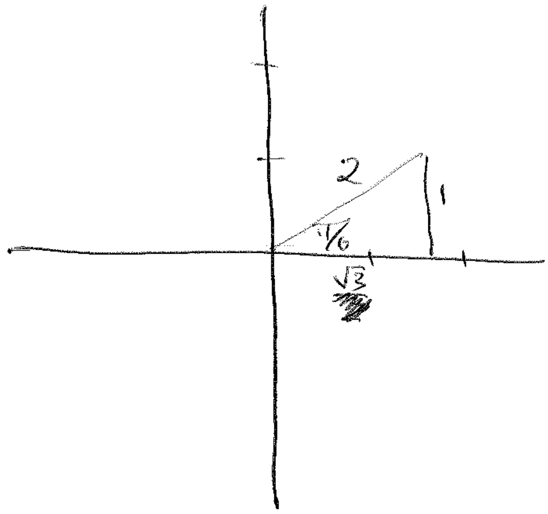
Q6 a) $3 - 3i$

$= 3\sqrt{2}\ \text{cis}\left(-\pi/4\right)$



b) $2\ \text{cis}\ \pi/6$

$= \sqrt{3} + i$



Q7 a) $\left(\text{cis}\,\pi/3\right)^{20} \div \left(2\,\text{cis}\,3\pi/4\right)$

$= \text{cis}\,20\pi/3 \div 2\,\text{cis}\,3\pi/4$

$= \text{cis}\,2\pi/3 \div 2\,\text{cis}\,3\pi/4$

$= \tfrac{1}{2}\,\text{cis}\left(2\pi/3 - 3\pi/4\right)$

$= \tfrac{1}{2}\,\text{cis}\left(\left(\frac{8}{12} - \frac{9}{12}\right)\pi\right)$

$= \tfrac{1}{2}\,\text{cis}\left(-\pi/12\right)$

b) $\sqrt{4\,\text{cis}\,\pi}$

$= \left(4\,\text{cis}\,\pi\right)^{1/2}$

$= 4^{1/2}\,\text{cis}\,\pi/2$

$= 2\,\text{cis}\,\pi/2$

$= 2i$

Q8 $A = \begin{pmatrix} 4 & -2 \\ a & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}$

a) $3A = 3\begin{pmatrix} 4 & -2 \\ a & 1 \end{pmatrix}$

$= \begin{pmatrix} 12 & -6 \\ 3a & 3 \end{pmatrix}$

b) $A + B = \begin{pmatrix} 4 & -2 \\ a & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}$

$= \begin{pmatrix} 6 & -2 \\ a-5 & 2 \end{pmatrix}$

c) $AB = \begin{pmatrix} 4 & -2 \\ a & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}$

$= \begin{pmatrix} 18 & -2 \\ 2a-5 & 1 \end{pmatrix}$

d) $A' = \begin{pmatrix} 4 & -2 \\ a & 1 \end{pmatrix}'$

$= \begin{pmatrix} 4 & a \\ -2 & 1 \end{pmatrix}$

Q8 cont'd

e) $|A| = \begin{vmatrix} 4 & -2 \\ a & 1 \end{vmatrix}$

$= 4 + 2a$

f) $A^{-1} = \frac{1}{4+2a}\begin{pmatrix} 1 & 2 \\ -a & 4 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{4+2a} & \frac{2}{4+2a} \\ \frac{-a}{4+2a} & \frac{4}{4+2a} \end{pmatrix}$ [optional step]

Q1

Let $\sqrt{i} = a+bi$

$i = (a+bi)^2$

$i = a^2 + 2abi - b^2$

Equating real parts

$a^2 - b^2 = 0$ ..... ①

Equating imaginary parts

$2ab = 1$ .... ②

② $\Rightarrow$ $b = \frac{1}{2a}$ .... ③

Sub ③ into ① $\Rightarrow$ $a^2 - \frac{1}{4a^2} = 0$ ... ④

④ $\Rightarrow$ $a^2 = \frac{1}{4a^2}$

$a^4 = \frac{1}{4}$

$a = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$

if $a = \sqrt{2}$ $b = \frac{1}{2a} = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$

if $a = -\sqrt{2}$ $b = \frac{1}{2a} = \frac{-2}{2\sqrt{2}} = \frac{-\sqrt{2}}{2}$

$\therefore \sqrt{i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ [or $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$] — not required

Q2 RTP $\overline{wz} = \bar{w}\,\bar{z}$

Let $w = a+bi$, $z = c+di$

$$
\begin{aligned}
LHS &= \overline{wz} \\
&= \overline{(a+bi)(c+di)} \\
&= \overline{ac \mp bd + (ad+bc)i} \\
&= ac - bd - (ad+bc)i
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \overline{a+bi} \times \overline{c+di} \\
&= (a-bi) \times (c-di) \\
&= ac - bd - (ad+bc)i \\
&= LHS
\end{aligned}
$$

QED

Q3 $\begin{pmatrix} 2 & 3 & 0 \\ -1 & 5 & 1 \end{pmatrix} + 2X = 3\begin{pmatrix} 0 & 1 & 0 \\ 4 & 2 & 5 \end{pmatrix}$

X must be 2×3

$$2X = \begin{pmatrix} 0 & 3 & 0 \\ 12 & 6 & 15 \end{pmatrix} - \begin{pmatrix} 2 & 3 & 0 \\ -1 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 13 & 1 & 14 \end{pmatrix}$$

$$X = \begin{pmatrix} -1 & 0 & 0 \\ 6\frac{1}{2} & \frac{1}{2} & 7 \end{pmatrix}$$

Q4 A^{-1} does not exist if $4 \times 1 - (-2) \times a = 0$

ie $4 + 2a = 0$

$2a = -4$

$a = -2$

So A^{-1} ~~do~~ exists for all a except -2

Q5 $A = \begin{pmatrix} 5 & a \\ -10 & -4 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 5 & a \\ -10 & -4 \end{pmatrix}\begin{pmatrix} 5 & a \\ -10 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 25-10a & a \\ -10 & 16-10a \end{pmatrix}$$

$A^2 = A$

$$\therefore \begin{pmatrix} 25-10a & a \\ -10 & 16-10a \end{pmatrix} = \begin{pmatrix} 5 & a \\ -10 & -4 \end{pmatrix}$$

So $25-10a = 5$

$20 = 10a$

$a = 2$

Check with $16-10a = -4$

$16-20 = -4$ ✓

Q6 Let the cost of a maths book be m
. . . . . physics b . . . p
. . . . . chemistry . . . c

$80m + 30p + 45c = 8505$
$50m + 10p + 20c = 4190$
$66m + 27p + 22c = 6256$

$$\begin{pmatrix} 80 & 30 & 45 \\ 50 & 10 & 20 \\ 66 & 27 & 22 \end{pmatrix}\begin{pmatrix} m \\ p \\ c \end{pmatrix} = \begin{pmatrix} 8505 \\ 4190 \\ 6256 \end{pmatrix}$$

$$\begin{pmatrix} m \\ p \\ c \end{pmatrix} = \begin{pmatrix} 80 & 30 & 45 \\ 50 & 10 & 20 \\ 66 & 27 & 22 \end{pmatrix}^{-1}\begin{pmatrix} 8505 \\ 4190 \\ 6256 \end{pmatrix}$$

$$\begin{pmatrix} m \\ p \\ c \end{pmatrix} = \begin{pmatrix} 45 \\ 72 \\ 61 \end{pmatrix}$$

Cost for Spannerville

$= 56 \times 45 + 29 \times 72 + 19 \times 61$

$= \$5767$

Q7 $D =$

	A	B	C	D
A	0	1	0	1
B	0	0	1	0
C	1	0	0	1
D	0	1	0	0

$$= \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad V_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

This is insufficient to rank all the players, so we will use D^2

$$D^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

$$V_1 + V_2 = \begin{pmatrix} 4 \\ 3 \\ 5 \\ 2 \end{pmatrix}$$

~~Annie and~~ So the ranking is Carla, Annie, Bob, Dog

Q8 $$\frac{1}{2\sqrt{5}+\sqrt{15}+2\sqrt{3}+3}$$

$$= \frac{1}{(\sqrt{3}+\sqrt{5})(2+\sqrt{3})}$$

$$= \frac{1}{(\sqrt{3}+\sqrt{5})} \times \frac{1}{2+\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}+\sqrt{5}} \times \frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}} \times \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$= \frac{\sqrt{3}-\sqrt{5}}{-2} \times \frac{2-\sqrt{3}}{1}$$

This will do. Further possible simplification:-

$$= \frac{(\sqrt{5}-\sqrt{3}) \times (2-\sqrt{3})}{2}$$

$$= \frac{2\sqrt{5}-\sqrt{15}-2\sqrt{3}+3}{2}$$