

A Practical Problem with an Easy Solution Using Complex Numbers



Two pirates landed on a small deserted island to bury some treasure. On the island were just three trees – a palm tree, a casuarina and a rather dead-looking hibiscus sort of thing. Otherwise the ground was fairly featureless grassy sand. They knew that other pirates might come looking for the treasure, so they decided to use a fairly obscure method to position it. Both women started at the hibiscus bush. The first one paced to the palm trees, then turned 90° right and paced the same distance again. The second one paced to the casuarina tree, turned 90° left, then paced the same distance again. They then buried the treasure half way between the points they ended up at.

Nine years later, the two pirates returned to the island to dig up the treasure. Because of the action of tides and storms, the island had changed shape a bit. The two palm trees were still there, but there was no trace of the hibiscus. 'How on earth will we find it now?' exclaimed Alice. 'I'm not digging up the whole *@~#★ island!'

'We won't have to,' replied Slasher. 'If we start at the palm tree, walk half way to the casuarina, then turn left and walk the same distance again, the treasure will be there.'

'But that's not where we buried it,' said Alice. 'Remember we started from that hibiscus bush which ain't here no more.'

'Arrrgh!' said Slasher, 'but the new directions get us to the same spot.'

'How do you know?' said Alice.

'I worked it out with complex numbers the night before we raided that Portuguese galleon. Look, I'll show you.'

Slasher was right. Here is her proof with a few crucial bits missing.

Consider the complex plane with its origin (O) at the hibiscus bush. P and C are the palm and casuarina respectively, Q and D are the points the pirates paced to. T is the midpoint of QD, the point where they buried the treasure. A is the mid-point of PC, B the point where Slasher claims the treasure can be found.

Let the points P and C be represented by the complex numbers u and v respectively.

Then $Q = u - im$ and $D =$

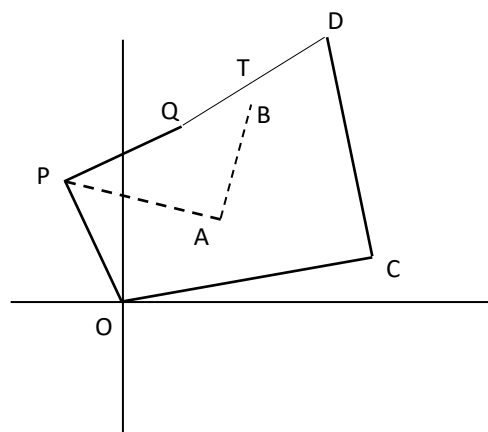
$T =$

$A =$

$B =$

$=$

$= T$



Your job is to rewrite the last four lines with the missing parts replaced.

A Solution

Let the points P and C be represented by the complex numbers u and v respectively.

Then $Q = u + iv$ and $D = v - iv$

$$T = \frac{1}{2}(u + iv + v - iv)$$

$$A = \frac{1}{2}(u + v)$$

$$B = \frac{1}{2}(u + v) - \frac{1}{2}i(v - u) = \frac{1}{2}(u + iv + v - iv) = T$$

So the place she claims the treasure can be found is the place where it is buried.

Out of interest, here is a vector solution to the same problem

Let \vec{OP} be $a\mathbf{i} + b\mathbf{j}$ and \vec{OR} be $c\mathbf{i} + d\mathbf{j}$

Then $\vec{PQ} = b\mathbf{i} - a\mathbf{j}$ and $\vec{RS} = \mathbf{j} - d\mathbf{i}$

$$\vec{OQ} = \vec{OP} + \vec{PQ} = a\mathbf{i} + b\mathbf{j} + b\mathbf{i} - a\mathbf{j}$$

$$\vec{OS} = \vec{OR} + \vec{RS} = c\mathbf{i} + d\mathbf{j} + \mathbf{j} - d\mathbf{i}$$

$$\begin{aligned} \vec{OT} &= \frac{1}{2}(\vec{OQ} + \vec{OS}) = \frac{1}{2}(a\mathbf{i} + b\mathbf{j} + b\mathbf{i} - a\mathbf{j} + c\mathbf{i} + d\mathbf{j} + \mathbf{j} - d\mathbf{i}) \\ &= \frac{1}{2}[(a + b + c - d)\mathbf{i} + (-a + b + c + d)\mathbf{j}] \end{aligned}$$

$$\vec{OY} = \frac{1}{2}(\vec{OP} + \vec{OR}) = \frac{1}{2}(a\mathbf{i} + b\mathbf{j} + c\mathbf{i} + d\mathbf{j})$$

$$\vec{PR} = \vec{OR} - \vec{OP} = c\mathbf{i} + d\mathbf{j} - a\mathbf{i} - b\mathbf{j}$$

$$\vec{PY} = \frac{1}{2}\vec{PR} = \frac{1}{2}(c\mathbf{i} + d\mathbf{j} - a\mathbf{i} - b\mathbf{j}) = \frac{1}{2}[(c - a)\mathbf{i} + (d - b)\mathbf{j}]$$

$$\vec{YZ} = \frac{1}{2}[(c - a)\mathbf{j} + (b - d)\mathbf{i}] = \frac{1}{2}(c\mathbf{j} - a\mathbf{j} + b\mathbf{i} - d\mathbf{i})$$

$$\begin{aligned} \vec{OZ} &= \vec{OY} + \vec{YZ} = \frac{1}{2}(a\mathbf{i} + b\mathbf{j} + c\mathbf{i} + d\mathbf{j} + c\mathbf{j} - a\mathbf{j} + b\mathbf{i} - d\mathbf{i}) \\ &= \frac{1}{2}[(a + b + c - d)\mathbf{i} + (-a + b + c + d)\mathbf{j}] = \vec{OT} \end{aligned}$$

So the place she claims the treasure can be found is the place where it is buried.

Geometric and coordinate solutions seem to be more difficult.