Numbers Beyond Our Imagination

> *Note: The name "googol" was invented by a young nephew of the American mathematician Edward Kasner when the boy was asked to make up a name for a very large number.*

What is the largest number you can think of? A billion? A trillion? If you wrote down a one and started adding zeros. it wouldn't take you very long to write a number so big that it would be beyond the names for large numbers that even a mathematician knows. Schroeder has thought of a googol which, believe it or not, is the accepted mathematical name for the number 1 followed by a hundred zeros! Although it has a name, this number is far too big for anyone to be able to understand it. In fact, it is dif1icult for us to appreciate the size of a number even as "small" as a million, since we have had no personal experience with such a number.

Do you know how large a million is'? It is easy to write 1.000,000, but how big is that? If you counted to one million and were able to name a number every second without stopping, it would take you nearly 12 days! And how long is a million days? A million days ago was in the 8th century B.C. How far is a million inches" Almost 16 miles.

If our idea of the size of a million is fuzzy, our notions of larger numbers must be even fuzzier. To take another example. how large is a billion? Oddly enough It depends upon Where you live! In the United States a billion is 1,000,000,000, while in England it is 1,000,000,000,000. Perhaps the reason for this difference is that until recently in history there was no need for such a large number and so everyone has not yet agreed on how to name it. Using the smaller version, a billion seconds is still a long time; a billion seconds from now is in the beginning of the 21 st century.

Is there much difference in the size of two numbers such as

100000000 and 1000000000?

At a glance they look about the same, yet when we add a zero to the first number to give the second, we have multiplied it by ten. Don't be fooled into thinking that since zero equals nothing, adding a zero to a number doesn't make much difference.

Each number in the sequence

1 10 100 1,000 10,000 100,000 1,000,000 . . .

is ten times larger than the number before. Another way to write this sequence is

$1\ 10\ 10^2\ 10^3\ 10^4\ 10^5\ 10^6$ . . ,

and this suggests that using exponents is a compact way of writing large numbers. Notice that the *exponent* of each 10 is also the *number of zeros* that follow the 1 if the number is written the long way. This pattern suggests that it makes sense to write 10 as 10^1 and even 1 as 10° (1 followed by 0 zeros).

The beginning of the sequence can then be written as

$10^0\ 10^1\ 10^2\ 10^3$ . . . .

Here is a table listing the names and exponential forms of some large numbers.

10^2	hundred	10^{18}	quintillion
10^3	thousand	10^{21}	sextillion
10^6	million	10^{24}	septillion
10^9	billion	10^{27}	octillion
10^{12}	trillion	$10^{3○}$	nonillion
10^{15}	quadrillion	10^{33}	decillion

Exercise Set I

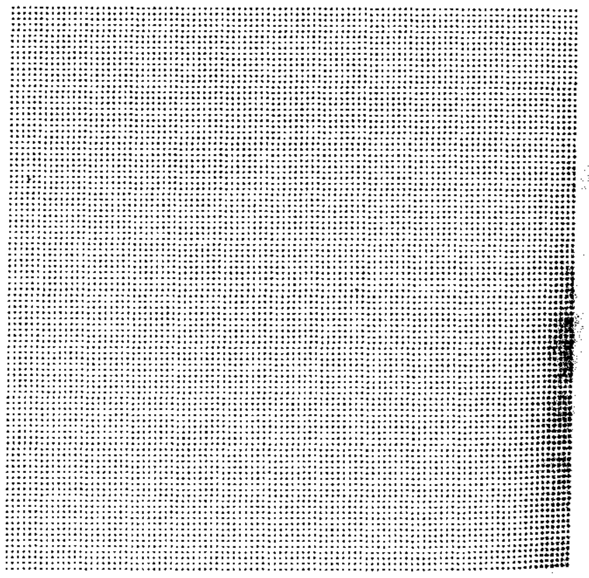
1. How many zeros follow the 1 if the number one quadrillion is written the long way?

2. Write in exponential form:
   a) ten million
   b) one hundred septillion,
   c) one thousand decillion.

3. Write a googol, using the exponential form.

4. What does 10° mean?

5. Guess, without counting anything, how many dots are in this figure. Write your guess as a power of 10.



6. The human brain contains about 10^{10} cells, called neurons and is far more complex than any computer that has been built. Write the number of cells the long way and then name the number in words. If this number was represented by a line of dots like this, it would stretch more than half way around the earth!

7. A tourist from England visited Yankee Stadium to see a baseball game. He didn't understand the game at all and left when the scoreboard read:
   1 0 0 0 0 0 0 0 0
   1 0 0 0 1 0 0 0 0
   When a boy outside the gate asked him the score, he answered: "It's up in the millions." How did the man get so confused?

8. Our solar system is part of the Milky Way galaxy, which is estimated to contain more than one hundred billion stars. Write this number in two different ways.

9. Suppose only one star out of every million was the sun of a planet with intelligent life on it. Approximately how many planets with intelligent life would there be in the Milky Way?

10. Water molecules are so small that if you could pour a *million* of them into a quart bottle every second and never stop until the bottle was full, it would take more than 10^8 years.

If you poured at the rate of a *billion* per second instead, how long would it take? (Since you are pouring at a faster rate, of course it would take less time.)

11. How long at the rate of a *trillion* per second?

You know that 10^3 means "1 followed by 3 zeros," or 1,000. Don't assume, however, that this is true if the number is written as a power of a number other than 10. For example, to find out what 100^3 means, you should think: $100^3 = 100 \times 100 \times 100 = 1,000,000$.

12. Which of these numbers is largest? 100^4 or $1,000^3$ or $10,000^2$ Write each number the long way and see.

Notice that $100^4 = (10^2)^4$ and that $100^4 = 10^8$, since you have just written 100^4 as 100,000,000. So $100^4 = (10^2)^4 = 10^{2x4} = 10^8$.

13. Now use this method to change $1,000^3$ and $10,000^2$ to powers of 10.

14. Can you express a googol squared as a power of 10? 15. How about a googol cubed?

Exercise Set II

The Greek mathematician Archimedes, who lived in the third century B.C., wrote a book called *The Sand Reckoner* in which he devised a method for forming large numbers.

l. The Greek name for 10,000 was "myriad." Write this number, using an exponent.

2. Archimedes began by thinking of a "myriad of myriads." What did he mean by this?

3. What would we call this number?

4. Write it in exponential form.

5. Archimedes called this number an "octade." Can you tell why? (Hint: What do an octet and an octopus have in common?)

6. He named the number 1016 "the second octade." What is our name for this number?

Notice that $10^{16} = (10^8)^2$, so that "the second octade" could also be called "an octade squared."

7. What number do you suppose Archimedes meant by "the third octade"? Write it as a power of 10 and give our name for this number.

8. An extraordinarily large number thought of by Archimedes was "an octade of octades," which we would write as $10^{800,0000.000}$

   a) If this number were to be written out in full, what would you have to do?

      Notice that it may be quite easy to *write* a large number such as a million in full: 1,000,000; it is *not* easy, however, to count from one to a million.

   b) Do you think that a computer could *print out* Archimedes' "octade of octades" in the long form?

   c) Do you think it could *count up* to this number?

Exercise Set III

A telephone cable across the Atlantic Ocean connects Newfoundland with Scotland. To keep the sound loud enough to hear, there are 51 amplifiers spaced fairly evenly along its length. Each one of these amplifiers increases the signal strength by about a million times to make up for fading of the signal along the cable. The first amplifier on the way from Newfoundland to Scotland strengthens the signal 10^6 times. After it passes through the second amplifier, it has been boosted 10^6 x 10^6 times.

1. How many times is that: 10^{12} or 10^{36}? Write the numbers out and see.

2. Upon passing through the third amplifier, the signal has been strengthened 10^6 x 10^6 x 10^6 or $(10^6)^3$ times. Altogether how many times has it been boosted by the time it passes through the 51st amplifier just before reaching Scotland?

The Mathematics of Large Numbers

SCIENTISTS in many fields need to write large numbers and make calculations with them. If these numbers are written the "long way," they are awkward to work with and it is easy to make a mistake in putting down the proper number of zeros. Here are some examples. The astronomer uses a unit of distance called a light-year, which is about 5,880,000,000,000 miles. The physicist has found that x-rays have frequencies of more than 10,000,000,000,000,000 vibrations per second. The chemist can figure out that an ounce of gold contains approximately 8,650,000,000,000,000,000,000 atoms.

You have already had some experience with writing numbers in a compact way by using exponents. Let's see how these numbers from science could be written in a more convenient form.

Peter says that the sun is "93 million miles high"; we can write this number as 93,000,000, or, since

$$93{,}000{,}000 = 93 \times 1{,}000{,}000$$

as

$$93 \times 10^6.$$

It is also true that

$$93{,}000{,}000 = 9.3 \times 10{,}000{,}000,$$

so the sun's "long fall" can be written as

$$9.3 \times 10^7$$

miles. When we write it *this* way, the number is said to be in *scientific notation.*

When a number is written as some number, which is at least 1 but less than 10, multiplied by 10 raised to some power, it is expressed in scientific notation.

Compare 93,000,000 and 9.3×10^7. In the scientific form, the decimal point has been moved from the end of the number to just after its first digit. The point has been moved 7 decimal places and the 7 becomes the exponent of the 10.

$$9\,3,0\,0\,0,0\,0\,0 = 9.3 \times 10^7$$

7 places

Following the same procedure, the astronomer's "light-year" number, 5,880,000,000,000, can be written as 5.88×10^{12}; the physicist's "x-ray" number, 10,000,000,000,000,000, can be written as 1×10^{16}; and the chemist's "gold atom" number, 8,650,000,000,000,000,000,000, can be written as 8.65×10^{21}. It is easy to see that the larger the number, the greater the advantage of writing it in scientific notation.

The biologist knows that a single red cell of human blood contains 2.7×10^8 hemoglobin molecules. How many is that? We can do what we did before, but in reverse, to find out.

$$2.7 \times 10^8 = 2.7 \times 100000000 = 270000000$$

Writing in a couple of commas, we have 270,000,000, or two hundred and seventy million hemoglobin molecules in one red cell!

Exercise Set IV

1. The earth picks up approximately 2.5×10^7 pounds of dust from the sky each day. Write this number in the long form and then name the amount in words.

2. It would take 3,000,000,000,000,000,000,000,000,000 candles to give as much light as the sun. Write this number in scientific notation.

3. The number of grains of sand in Malibu Beach, California, is about 1×10^{14} Write this number in the long form and then name the amount in words.

4. The weight of the water in all of the oceans of the world is 1,580,000,000,000,000,000 tons. Write this number in scientific notation.

5. The number of different hands which it is possible for you to be dealt in a game of bridge is about 6.35×10^{11}. Write this number in the long form and then name it in words.

6. The total number of grains of wheat for which the inventor of chess is said to have asked the King of Persia almost 18,450,000,000,000,000,000. Write this number in scientific notation.

7. In 1626, Peter Minuit paid the Indians of New York $24 for Manhattan Island. If today he completely covered Manhattan with dollar bills, it would take more than 5.94×10^9 of them and this would not be nearly enough to pay for the island (including its property)! How much money is this? Name the amount in words.

8. Proxima Centauri, the nearest star beyond the sun, is 25 trillion miles away. One way of writing this distance is 25×10^{12} miles, but this number is not in scientific notation. Why not? Rewrite it in scientific notation. (Hint: $25=2.5 \times 10$; $10 \times 10^{12}=$?)

10. Inflation of the value of money is a serious economic problem. In 1946, inflation of the currency was so bad in Hungary that the gold pengo was worth 130 quintillion paper pengos. Write this number in scientific notation. (The pengo was replaced that year by another unit of money.)

Exercise Set V

One way to multiply two numbers written in scientific notation is to change them to their long forms before multiplying them. For example,

$$(2 \times 10^2) \times (3 \times 10^3) = 200 \times 3{,}000 = 600{,}000.$$

Notice that the result in scientific notation is 6×10^5. We can also get that answer in this way:

$2 \times 10^2 \times 3 \times 10^3 = 2 \times 3 \times 10^2 \times 10^3 = 6 \times 10^5$

1. When powers of 10 are multiplied, are the exponents *multiplied* or *added?*

2. Multiply the two numbers (4×10^2) and (2×10^4) by first changing them to their long forms. Then change the result to scientific notation. Does it agree with your answer to the previous question about exponents?

3. Copy and complete the following statement: "To multiply two numbers in scientific notation, ____ the numbers in front and ______ the exponents."

   Notice that in this problem,

   $$(5 \times 10^3) \times (6 \times 10^4) = 30 \times 10^7,$$

   the result is not in scientific notation, but

   $$30 \times 10^7 = 3 \times 10 \times 10^7 = 3 \times 10^8.$$

   Write the results in each of the following problems in scientific notation.

4. $(8 \times 10^2) \times (5 \times 10^6)$

5. $(3 \times 10^{11}) \times (7 \times 10^5)$
6. $(9 \times 10^3) \times (1.5 \times 10^{10})$

7. The earth travels about 5.8×10^8 miles in its trip around the sun each year. What distance does it travel around the sun in 1,000 years?

8. Our solar system is about 3×10^4 light-years from the center of the Milky Way galaxy. Assuming a light-year to be approximately 6×10^{12} miles, how far are we in miles from the center of the Milky Way?

9. During one summer, it is possible for a couple of house flies to become parents and ancestors of 1.9×10^{20} flies. Suppose the Swindle Swatter Company decides to improve its business prospects by breeding 5×10^9 pairs of house flies in strategic locations all over the country. How many flies could theoretically be raised during the summer?

10. Every minute more than 8.4×10^{11} drops of water flow over Niagara Falls. Each drop contains 1.7×10^{21} molecules. How many molecules of water pass over Niagara Falls in one minute?

Exercise Set VI

In one of their ads, the Volkswagen Company claims that *"exactly* 1,612,462 beans" can be put into a Volkswagen station wagon.

1. Write this number in scientific notation.

2. Is this way of writing the number more compact than writing it in the long form?

Suppose we round off the number of beans and say that a V.W. wagon will hold *about* 1,600,000 beans.

3. How does this number look in scientific notation?

4. Is this way of writing the number any shorter than writing it in the long form?

5. When a scientist deals with large numbers, do you think they are *exact* or *approximate?*

6. For which kind of numbers is scientific notation more appropriate: exact numbers or approximate numbers?

Exercise Set VII

1. The distance from the earth to the sun is about 140 000 000 km.
   a) Express this in scientific notation on your calculator screen.
   b) Convert it to centimetres.
   c) A $20 note is 16 cm long. How many $20 notes would fit in a straight line between the earth and the sun?

2. The distance light travels in a year is about 9 000 000 000 000 km. (This distance is called a light year.)
   a) Express this in scientific notation on your calculator screen.
   b) Alpha Centauri is 4 light years from Earth. What is this distance in kilometres?
   c) If you drove at 120 km/hr how many hours would it take you to drive to Alpha Centauri?
   d) If you flew a jet plane at 2000 km/hr, how many hours would it take you to fly to Alpha Centauri?
   e) How many years is this (Use $365^1/_4$ days a year)?

3. The formula for calculating the amount A that a given principal P grows to at an interest rate of i% per annum for n years is: $A = P(1 + i/100)^n$

   If your great-great-great-…-great uncle left $1 in the Bank of Jerusalem for you compounded annually at 2% per annum in the year 1 AD (is, 2006 years ago), what amount has that $1 grown to today?

4. Rubik's cube was a popular puzzle some years ago. The cube could be rotated and twisted in many different orientations - about 4.3×10^{19}ifferent orientations in fact. If you could rotate the cube to a different orientation each second, without duplication, how long would it take you to get every possible orientation of the cube ...
   a) in hours?
   b) in days?
   c) in months?
   d) in years?
   e) in generations (assume 1 generation = 35 years)?

f) in millennia (1 millennium = 1 000 years)?

5. Assume all grains of sand are congruent and 4 grains laid in a line would measure a millimetre. Approximately how many grains of sand would need to be placed in a row to reach from Rockhampton to Brisbane (a distance of about 600 km)?

6. There are 24 ways of arranging 4 people in a row. It is calculated this way: Any one of the 4 people can be plated on the left. For each of those choices, any of the remaining 3 can be placed next, then either of the remaining 2 and finally the remaining 1 person fills the spot on the right. Thus the total number of arrangements is 4 x 3 x 2 x 1 = 24. (List all the arrangements of the letters A, 8, C and D if you don't believe this.)

   At an upcoming International Trade meeting the 14 current members stand in a row for the traditional photograph. So that no one feels snubbed by being put on the end, it is decided that the members will be photographed in all possible arrangements.

   If it takes 10 seconds for each photograph, how long will it take to complete the photography session?