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Bretschneider's formula - Area of any convex quadrilateral

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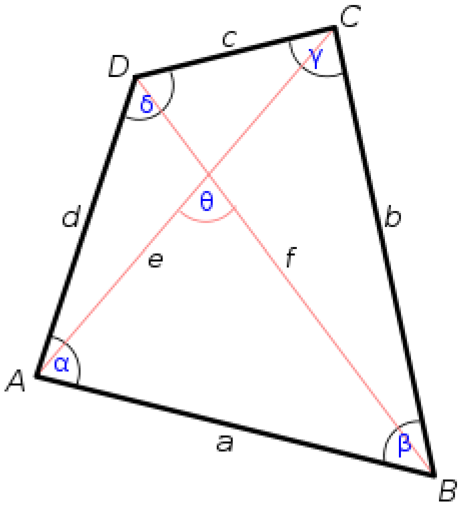

A quadrilateral.

In geometry, **Bretschneider's formula** is the following expression for the area of a general convex quadrilateral:

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cdot \cos^2\left(\frac{\alpha+\gamma}{2}\right)}.$$

Here, *a*, *b*, *c*, *d* are the sides of the quadrilateral, *s* is the semi-perimeter, and $\alpha$and $\gamma$are two opposite angles.

Bretschneider's formula works on any convex quadrilateral regardless of whether it is cyclic or not.

The German mathematician Carl Anton Bretschneider discovered the formula in 1842. The formula was also derived in the same year by the German mathematician Karl Georg Christian von Staudt.

Proof of Bretschneider's formula

Denote the area of the quadrilateral by *K*. Then we have

$$K = \text{area of} \triangle ADB + \text{area of} \triangle BDC$$
$$= \frac{ad\sin\alpha}{2} + \frac{bc\sin\gamma}{2}.$$

Therefore

$$4K^2 = (ad)^2\sin^2\alpha + (bc)^2\sin^2\gamma + 2abcd\sin\alpha\sin\gamma.$$

The Law of Cosines implies that

$$a^2 + d^2 - 2ad\cos\alpha = b^2 + c^2 - 2bc\cos\gamma,$$

because both sides equal the square of the length of the diagonal *BD*. This can be rewritten as

$$\frac{(a^2 + d^2 - b^2 - c^2)^2}{4} = (ad)^2\cos^2\alpha + (bc)^2\cos^2\gamma - 2abcd\cos\alpha\cos\gamma.$$

Substituting this in the above formula for $4K^2$ yields

$$4K^2 + \frac{(b^2 + c^2 - a^2 - d^2)^2}{4} = (ad)^2 + (bc)^2 - 2abcd\cdot\cos(\alpha + \gamma).$$

This can be written as

$$16K^2 = (a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a) - 16abcd\cdot\cos^2\left(\frac{\alpha + \gamma}{2}\right).$$

Introducing the semi-perimeter

$$s = \frac{a + b + c + d}{2},$$

the above becomes

$$16K^2 = 16(s-a)(s-b)(s-c)(s-d) - 16abcd\cdot\cos^2\left(\frac{\alpha+\gamma}{2}\right)$$

and Bretschneider's formula follows.