A proof of a statement is a piece of writing designed to help the reader be certain that the statement must be true

PROOFS

4 main techniques of proof are used in Mathematics:

- proof by exhaustion
- proof by contradiction
- proof by deduction
- proof by induction

Proof by Exhaustion

Every possibility must be tried. For instance, finding 140 Sri Lankans with 10 toes would not prove that all Sri Lankans have 10 toes. But checking all of them and seeing that they all do would constitute proof. Finding just one without 10 toes would constitute disproof.

To prove: The set {0, 1, -1} is closed under multiplication.

There are 9 possible results of multiplication on this set. These are given below:

x	0	1	-1
0	0	0	0
1	0	1	-1
-1	0	-1	1

All of these are elements of the set.
∴ the set is closed under multiplication. QED (**quod erat demonstrandum: what was to be proved**)

To prove: 13 is prime.

13÷2 = 6.5	13÷3 = 4.33..	13÷4 = 3.25	13÷5 = 2.6
13÷6 = 2.16..	13÷7 = 1.85..	13÷8 = 1.62..	13÷9 = 1.44..
13÷10 = 1.3	13÷11 = 1.18..	13÷12 = 1.08..	

∴ 13 is prime. QED

Proving that an infinite set such as the whole numbers is closed under multiplication would obviously not be possible by exhaustion, because you would have to test an infinite number of possibilities.

Proving that 1 000 000 001 is a prime number by this method would take a long time, though it is possible. It can be done fairly easily with the help of a computer.

Proof by Contradiction

To prove something, assume the opposite and show that this leads to a false conclusion.

To prove: The set of primes is infinite.

Assume that there is a finite number, k, of primes.
Name them $p_1, p_2, p_3, \ldots p_k$. Any number $>p_k$ is not prime.

Multiply all the primes together and add 1. Call the resulting number n.
Thus $n = p_1 x p_2 x \ldots x p_k + 1$
Obviously $n > p_k$
$\therefore$ n is not prime
$\therefore$ n divides evenly by at least one prime p_x.
p_x cannot be any of $p_1, p_2, \ldots p_k$ because all of these will leave a remainder 1 on division.
$\therefore \exists$ a prime p_x in addition to $p_1, p_2, \ldots p_k$.
$\therefore$ the assumption is false: there is not a finite number of primes. QED

To prove: √2 is irrational.

Assume √2 is rational.
Then we can express √2 as a fraction **in simplest form** a/b where a and b are integers.

$a/b = \sqrt{2}$
$\therefore a^2/b^2 = 2$
$\therefore a^2 = 2b^2$
$\therefore a^2$ is even
$\therefore$ a is even

So a can be written as 2k where k is an integer
Then $(2k)^2 = 2b^2$
ie. $4k^2 = 2b^2$
$\therefore 2k^2 = b^2$
$\therefore b^2$ is even
$\therefore$ b is even

ie. a and b are both even, thus a/b is not in simplest form.
Thus our assumption must be incorrect and √2 is irrational. QED

Proof by Deduction

There are 5 main layouts for proof by deduction.

1. In the first layout a statement is made which is obvious. From this a logical conclusion is drawn. From that a further logical conclusion is drawn and so on until the final conclusion is the statement to be proved.

ie. obvious statement
=> next statement
=> . . . . .
=> next statement
=> statement to be proved

The other 4 layouts can be used where we need to prove that a given mathematical expression equals a second mathematical expression. These expressions are called the Left Hand Side and the Right Hand Side.

2. The LHS is manipulated to a different form, then a further form and so on until we arrive at the expression on the RHS

ie. LHS = ...
= . . . .
= . . . .
= RHS

3. Ditto, working the other way

ie. RHS = ...
= . . . .
= . . . .
= LHS

4. The LHS is manipulated to a different (usually simpler) form, then the RHS is manipulated to the same form

ie. LHS = ...
= . . . .
= xxxx
RHS = ...
= . . . .
= . . . .
= xxxx
= LHS

5. The difference between the LHS and the RHS is proved to be 0

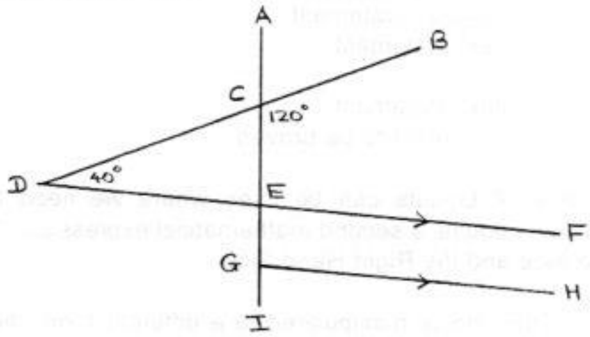
ie. LHS - RHS = ...
= . . . .
= . . . .
= 0

Examples

Layout 1. To prove: The squares of all odd numbers are odd.

All odd numbers can be written 2n+1 where n is a whole number.
The square of an odd number is, therefore $(2n+1)^2$, which equals $4n^2+4n+1$.
This can be written as $2(2n^2+2n)+1$, which must be an odd number. QED.

Layout 1. To prove: ∠HGI = 80°



∠BCE = 120° (given)
∴ ∠DCE = 60° (supplementary)
∠CDE = 40° (given)
∴ ∠CED = 80° (angles of a triangle)
∴ ∠FEG = 80° (vertically opposite)
∴ ∠HGI = 80° (corresponding) QED.

Layout 2. To prove: $\sqrt{\dfrac{(a^2-b^2)(a+b)}{a-b}} = a+b$

$$\begin{aligned}
\text{LHS} &= \sqrt{\frac{(a^2-b^2)(a+b)}{a-b}} \\
&= \sqrt{\frac{(a-b)(a+b)(a+b)}{a-b}} \\
&= \sqrt{(a+b)^2} \\
&= a+b \\
&= \text{RHS} \quad \text{QED}
\end{aligned}$$

Layout 3. To prove: $\sqrt{\dfrac{(a^2-b^2)(a+b)}{a-b}} = a+b$

$$\begin{aligned}
\text{RHS} &= a+b \\
&= \sqrt{(a+b)^2} \\
&= \sqrt{\frac{(a-b)(a+b)(a+b)}{a-b}} \\
&= \sqrt{\frac{(a^2-b^2)(a+b)}{a-b}} \\
&= \text{LHS} \quad \text{QED}
\end{aligned}$$

Layout 4. **To prove:** $\sqrt{\frac{(a^2-b^2)(a+b)}{a-b}} = 3(a+b)+c-(2a+2b+c)$

LHS $= \sqrt{\frac{(a^2-b^2)(a+b)}{a-b}}$

$= \sqrt{\frac{(a-b)(a+b)(a+b)}{a-b}}$

$= \sqrt{(a+b)^2}$

$= a+b$

RHS $= 3(a+b)+c-(2a+2b+c)$

$= 3a+3b+c-2a-2b-c$

$= a+b$

= LHS QED

Layout 5. **To prove:** $\sqrt{\frac{(a^2-b^2)(a+b)}{a-b}} = 3(a+b)+c-(2a+2b+c)$

LHS - RHS $= \sqrt{\frac{(a^2-b^2)(a+b)}{a-b}} - [3(a+b)+c-(2a+2b+c)]$

$= \sqrt{\frac{(a-b)(a+b)(a+b)}{a-b}} -[3a+3b+c-2a-2b-c]$

$= \sqrt{(a+b)^2} \quad -3a-3b-c+2a+2b+c$

$= a+b \ -3a-3b-c+2a+2b+c$

$= 0$ QED

Proof by Induction

Most commonly used for proving the correctness of formulae which have, as the independent variable, n where n is any whole number.

Assume the formula to be correct for one value of n. From this deduce that it is also true for the value 1 higher. As this conclusion is true whatever the value of n, then, if the formula is correct for any value of n, it is correct for the next value, and if it is correct for the next value, then it is also correct for the one after and so on.

So all that remains to be done to prove that it is correct for any n is to show that it is correct for n = 1.

To prove: The sum of the first n whole numbers is $\frac{1}{2}n(n+1)$.

Assume the formula to be correct for n = some number, k

Then the sum of the first k whole numbers is $\frac{1}{2}k(k+1)$

The sum of the first k+1 whole numbers would then be k+1 more than this,

ie. $\frac{1}{2}k(k+1) + k+1$

ie. $\frac{1}{2}k^2 + \frac{1}{2}k + k + 1$

ie. $\frac{1}{2}(k^2+k+2k+2)$

ie. $\frac{1}{2}(k^2+3k+2)$

ie. $\frac{1}{2}(k+1)(k+2)$.

Using the formula for n = k+1 gives the same result.

Thus if we assume the formula to be correct for n = k, then it is also correct for n = k+1.

The sum of the first 1 whole number is obviously 1.
The formula gives $\frac{1}{2} \times 1 \times (1+1) = 1$.

Thus the formula is correct for all n.

Prove that: if the statement is true for n = k,
then it is true for n = k+1

An Example of Proof by Induction

1	=1
1+3	=4
1+3+5	=9
1+3+5+7	=16
1+3+5+7+9	=25 . . . .

What is the sum of the first 100 odd numbers? It seems that it will be 100^2.
What is the sum of the first n odd numbers? It seems that it will be n^2.

Can we prove that this formula will always work?
As it is a formula and n can be any whole number, we will use induction.

To prove: The sum of the first n odd numbers = n^2.

Part 1: The sum of the first 1 odd number is obviously 1.
When n=1, the formula gives $1^2 = 1$.
So the formula is correct for n=1.

Part 2: a) Assuming the formula is correct for n=k, the sum of the first k odd numbers is k^2.
The $(k+1)^{th}$ odd number is 2k+1
So the sum of the first k+1 odd numbers is $k^2 + 2k+1$

b) The formula predicts $(k+1)^2$, which expands to k^2+2k+1

So whatever the value for k, the formula gives the correct result for k+1

The formula is true for n=1, so it is true for n=2
As it is true for n=2, it is true for n=3
As it is true for n=3, it is true for n=4 and so on.

So the formula is always true.

Summary of Proof by Induction

Part 1: Prove the formula true for n=1

Part 2: Prove that the fact that the formula is true for one number means that it must be true for the next number.

a) Assume the formula is true for n=k and hence work out the result for n=k+1.
b) See if the formula gives the correct result for n=k+1.

Exercises

1. Prove by induction that the sum of the first n even numbers is n(n+1).

2. Prove by induction that the sum of the first n square numbers is $^1/_6 n(n+1)(2n+1)$.

① Prove by exhaustion that the set $\{0, 1, 2, 3, 4\}$ under addition has an identity and an inverse for every element

② Prove (by exhaustion) that $x^2 - 2x > 2x$ for all integer values of x from 5 to 10 inclusive

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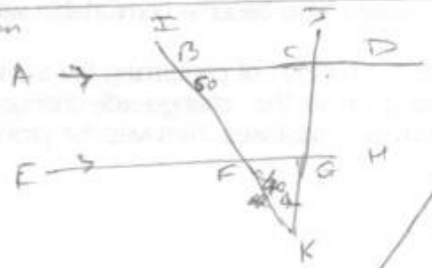
① Prove by contradiction that $\sqrt{3}$ is irrational

② ~~Prove that by contradiction that all odd numbers perfect squares have an odd number of factors~~

~~⓪ Prove squares of even numbers are even~~

① Prove by deduction that

$GCD = 100°$



② Prove by deduction that $\dfrac{1}{\frac{1}{x+1} \div \frac{1}{x-1}} = \dfrac{1}{\frac{1}{2}x^2 - \frac{1}{2}}$ $\frac{1}{2}\frac{x^2}{2} - \frac{1}{2}$

Induction

$1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \dots \frac{1}{n(n+1)} = \frac{n}{n+1}$

~~$n(n+1) \div 2$ for all n~~

~~$n^2 - 1 \div 8$ for all n~~