Circular Motion

In the following questions assume that the acceleration due to gravity at the Earth's surface is 9.8 m/s^2 and that the radius of the Earth is 6400 km.

C1. Rudolf is standing on the edge of a horizontal round platform 6 m in diameter. The platform is rotating once every 10 s. Find

a. Rudolf's angular velocity
b. Rudolf's linear velocity
c. Rudolf's acceleration
d. the minimum coefficient of friction between Rudolf and the platform for Rudolf not to slide off
e. the shortest period of rotation for Rudolf not to slide off if the coefficient of friction is 0.4

C2. A 1.5 t car is driving around a bend with radius 50 m at 72 km/h. Find

a. the acceleration of the car
b. the force required to provide this acceleration
c. the minimum coefficient of friction between the car tyres and the road if the road is level

C3. What is the greatest speed at which a car can drive round a 300 m circular track if the track surface is level and the coefficient of friction between the car's tyres and the track is 0.5.

C4. Consider the Earth. Now answer the following questions.

a. Find the centripetal acceleration at the equator due to the Earth's rotation.
b. What would the length of the day have to reduce to for the centripetal acceleration to be equal to the acceleration due to gravity?
c. What major assumption needs to be made to answer part b?
d. What would be some consequences for life on Earth if, over a few months, the Earth did speed up as calculated and this assumption was met?
e. What would be some consequences for life on Earth if it sped up as calculated and this assumption was not met?

C5. A neutron star has a mass one million times that of the Earth and a radius of 16 km. It spins at 60 revolutions per second. Find

a. its angular velocity
b. the linear velocity of the surface at the equator
c. the centripetal acceleration of the surface at the equator
d. the acceleration due to gravity at the surface at the equator
e. whether gravity is sufficient to provide the required centripetal acceleration

C6. A geostationary satellite is one that orbits the Earth remaining always above the same point on the equator. At what height above the ground must geostationary satellites be deployed?

Banked Tracks

C7. A car drives around a 600 m icy circular track at 108 km/h. At what angle should the track be banked for there to be no tendency for the car to slip sideways?

C8. A 400 m circular velodrome track is banked at 9°. At what speed were the designers expecting the cyclists to ride?

C9. A curve on a railway track has a radius of 200 m and the tracks are 1.2 m apart. A 200 t trains goes round the curve at 90 km/h.

a. How much sideways force does the train exert on the tracks?
b. How much higher should the outer track be than the inner track to prevent sideways force on the tracks?

C10. A cycle track consists of a hemispherical bowl with diameter 30 m. A stunt rider is cycling around the track at a constant height and speed. Assuming she is perpendicular to the track, and that she does 12 laps per minute, at what speed is she cycling?

Simple Harmonic Motion

S1. A weight is hanging on the end of a long spring, 0.6 m above the floor. It is lifted 0.2 m, then let go. It then performs simple harmonic motion, completing one cycle every 4 seconds.

a. Find the period, frequency and angular velocity of the motion.
b. Find expressions for x, the weight's height above the floor, $\dot{x}$, its velocity and $\ddot{x}$, its acceleration, in terms of t, the time since it was released.
c. Find expressions for $\dot{x}$ and $\ddot{x}$ in terms of x.
d. Find x, $\dot{x}$ and $\ddot{x}$ when $t = 2.5$ s.
e. Find the speed and acceleration of the weight when it is 0.75 m above the floor.
f. Find how long it takes the weight to fall from 0.8 m above the floor to 0.5 m above the floor.
g. Find the weight's greatest speed and greatest acceleration.
h. Find the weight's height and speed when its acceleration is 0.1 m/s^2 upwards.

S2. A particle is performing SHM with a frequency of 120 cycles/min. Its speed when it is 5 cm from the mean position is 60 cm/s.

a. Find the amplitude of the motion.
b. Find the greatest speed and the greatest acceleration.
c. Find the speed and acceleration when it is 3 cm from the mean position.

d. Find the time taken to travel from 3 cm from the mean position to its maximum distance from the mean position.

e. Find the distance from the mean position when the speed is 30 cm/s.

S3. The equation $\ddot{x} = -4x$ describes the motion of bean performing SHM. If the greatest speed of the bean is 4, find the period and amplitude of the motion.

S4. The equation $\dot{x} = 8\sqrt{9 - x^2}$ describes the motion of parrot performing SHM.

a. Find the acceleration in terms of x.

b. Find the period, the amplitude and the greatest speed.

S5. The equation $x = 4 \cos 3t$ describes the motion of a refrigerator performing SHM.

a. Find the velocity and acceleration in terms of t.

b. Find the amplitude, angular velocity and period.

c. Find the greatest speed and the greatest acceleration.

d. Find the velocity and acceleration in terms of x.

S6. An encyclopedia is moving with SHM. When it is 1.2 m from the mean position, its acceleration is 2.4 m/s^2 and its speed is 2 m/s.

a. Find the period and the amplitude.

b. Find the speed at the mean position.

c. Find the speed and acceleration when it is 1.5 m from the mean position.

Answers

C1. a. 0.628 rad/s b. 1.88 m/s c. 1.18 m/s^2 d. 0.121 e. 5.5 s

C2. a. 8 m/s^2 b. 12 kN c. 0.816

C3. 55 km/h

C4. a. 0.0338 m/s^2 b. 84.6 min

C5. a. 377 rad/s b. 6 031 000 m/s c. 2 274 000 000 m/s^2 d. 1 568 000 000 000 m/s^2 e. yes

C6. 35 940 km

C7. 43.9°

C8. 35.8 km/h

C9. a. 625 000 N b. 36.5 cm

C10. 61.8 km/h

S1. a. $P = 4$ s, $f = 0.25$ Hz, $\omega = \pi/2$ b. $x = 0.6 + 0.2 \cos 0.5\pi t$, $\dot{x} = -0.1\pi \sin 0.5\pi t$, $\ddot{x} = -0.05\pi^2 \cos 0.5\pi t$ c. $\dot{x} = \pm 0.5\pi\sqrt{0.04 - (x - 0.6)^2}$, $\ddot{x} = -0.25\pi^2 (x - 0.6)$ d. $x = -0.1414$ m, $\dot{x} = 0.2221$ m/s, $\ddot{x} = 0.3490$ m/s^2 e. speed = 0.2078 m/s, acc = 0.3701 m/s^2 f. 1.333 s g. 0.314 m/s, 0.4935 m/s^2 h. height = 0.5595 m, speed = 0.3076 m/s

S2. a. 6.91 cm b. 86.88 cm/s, 1092 cm/s^2 c. 78.27 cm/s, 474 cm/s^2 d. 0.08926 s e. 6.488 cm

S3. a. period = π, amplitude = 1.273

S4. a. $-64\,x$ b. P = $\pi/4$, amplitude = 3, greatest speed = 24

S5. a. $\dot{x} = -12 \sin 3t$, $\ddot{x} = -36 \cos 3t$ b. amplitude = 4, $\omega = 3$, P= $2\pi/3$ c. 12, 36

S6. d. $\dot{x} = \pm 3\sqrt{16 - x^2}$, $\ddot{x} = -9x$

S7. a. P = $\sqrt{2}\pi$, amplitude = 1.855 b. 2.623 m/s c. 1.543 m/s, 3 m/s^2

Solutions

Circular Motion – Solutions

C1 (a) Period = 10

Frequency = 0.1

$\omega = 0.2\pi = 0.628$ rad/s

(b) $v = \omega r$

$= 0.628 \times 3$

$= 1.88$ m/s

(c) $a = \omega^2 r$

$= 0.628^2 \times 3$

$= 1.18\ ms^{-2}$

(d) If Rudolf's mass is m, then

$\frac{m\omega^2 r}{mg} = \mu$

$\mu = \frac{2.37}{9.8} = 0.121$

(e) If $\mu = 0.4$, then $\omega^2 r$ can be 0.4×9.8

$3\omega^2 = 3.92$

$\omega^2 = 1.31$

$\omega = 1.14$

Period $= \frac{2\pi}{\omega} = 5.50$ s

C2 (a) Acceleration $= \frac{v^2}{r}$

$v = 72 \div 3.6 = 20$

$a = \frac{20^2}{50} = 8$ m/s²

(b) Force $= ma = 1500 \times 8$

$= 12000$ N

(c) $Mg = 1500 \times 9.8$

$\mu = \frac{1500 \times 8}{1500 \times 9.8}$

$= 0.816$

C3 Circumference of track = 300 m

radius $= 300 \div 2\pi = 47.75$ m

Greatest centripetal force $= \frac{mv^2}{r} = mg \times 0.5$

$v^2 = rg \times 0.5$

$= 47.75 \times 9.8 \times 0.5$

$= 233.96$

$v = 15.3$ m/s $= 55.1$ km/h

C4 (a) Acceleration $= \omega^2 r$

$= \left(\frac{2\pi}{24 \times 3600}\right)^2 \times (6.4 \times 10^6)$

$= 0.0338$ m/s²

(b) $\omega^2 r = 9.8$

$\omega^2 = 9.8 \div (6.4 \times 10^6)$

$= 1.531 \times 10^{-6}$

$\omega = 1.237 \times 10^{-3}$

$P = \frac{2\pi}{\omega}$

$= 5078$ s $= 84.6$ min

(c) That the Earth remains the same shape

(d) The oceans and atmosphere would all move to the equatorial regions, then be thrown off into space

(e) The Earth would be reshaped by massive earthquakes which would probably destroy most life. Then much of the Earth would be thrown off into space

C5 (a) Frequency = 60

$\omega = 120\pi = 377$ rad/s

(b) Linear velocity $= \omega r$

$= 377 \times 16\,000$

$= 6\,031\,000$ m/s

(c) Centripetal acceleration

$= \omega^2 r$

$= 377^2 \times 16\,000$

$= 2\,274\,000\,000$ m/s²

(d) Gravitational acceleration

$= 9 \times 10^6 \times \left(\frac{6400}{16}\right)^2$

$= 1\,568\,000\,000\,000$ m/s²

(e) This would be quite sufficient

C6 The centripetal acceleration would have to equal the gravitational acceleration

Let the distance from the Earth's centre be x

$$\omega = \frac{2\pi}{24 \times 3600}$$

$$\omega^2 x = 9.8 \times \left(\frac{6.4\times10^6}{x}\right)^2$$

$$\omega^2 x^3 = 9.8 \times (6.4\times10^6)^2$$

$$\left(\frac{2\pi}{24\times3600}\right)^2 x^3 = 9.8 \times (6.4\times10^6)^2$$

$$x^3 = 9.8 \times (6.4\times10^6)^2 \times \frac{(24\times3600)^2}{4\pi^2}$$

$$= 7.598 \times 10^{22}$$

$$x = 42\,340\,000 \text{ m}$$

$$= 42\,340 \text{ km}$$

Height above surface

$$= 42\,340 - 6400$$

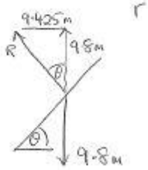
$$= 35\,940 \text{ km}$$

C7 Circumference of track = 600 m

Radius $= \frac{600}{2\pi} = 95.5$ m

Speed = 108 km/h = 30 m/s

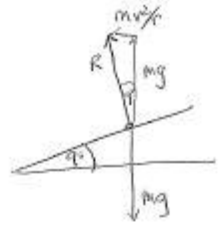
Centripetal acceleration $= \frac{v^2}{r} = \frac{900}{95.5} = 9.425$ m/s²



$$\tan\theta = \frac{9.425m}{9.8m} = 0.962$$

$$\theta = 43.9°$$

C8



Radius of track $= \frac{400}{2\pi} = 63.66$ m

$$\tan 9° = \frac{mv^2}{r} \div mg$$

$$= \frac{v^2}{rg}$$

$$= \frac{v^2}{63.66 \times 9.8}$$

$$v^2 = 63.66 \times 9.8 \times \tan 9°$$

$$= 98.81$$

$$v = 9.94 \text{ m/s} = 35.8 \text{ km/h}$$

C9 (a) Outward accⁿ $= \frac{v^2}{r}$

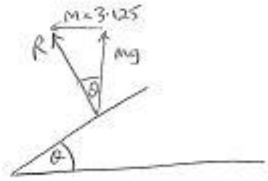
$$= \frac{25^2}{200}$$

$$= 3.125 \text{ m/s}^2$$

Outward force $= 2\times10^5 \times 3.125$ N

$$= 625\,000 \text{ N}$$

(b)



$$\tan\theta = \frac{3.125}{9.8} = 0.3189$$

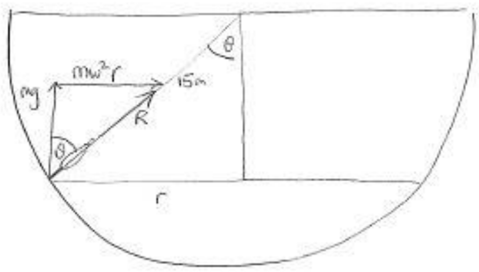
$$\theta = 17.69°$$

$$h = 1.2 \sin 17.69°$$

$$= 0.3646 \text{ m}$$

$$\simeq 365 \text{ mm}$$

C10



$$\tan\theta = \frac{m\omega^2 r}{mg} = \frac{\omega^2 r}{g}$$

Period = 5 s

$$\omega = \frac{2\pi}{5} = 1.2566 \text{ rad/s}$$

$$\therefore \tan\theta = \frac{1.2566^2 \times r}{9.8} = 0.1611\, r$$

$$\sin\theta = \frac{r}{15}$$

$$\cos\theta = \frac{\sin\theta}{\tan\theta} = \frac{r}{15} \div 0.1611\, r$$

$$= 0.4138$$

$$\theta = 65.55°$$

$$\therefore \tan\theta = 2.1999 = 0.1611\, r$$

$$\therefore r = \frac{2.1999}{0.1611} = 13.655$$

$$V = \omega r = \frac{2\pi}{5} \times 13.655$$

$$= 17.16 \text{ m/s}$$

$$= 61.8 \text{ km/h}$$

Simple Harmonic Motion - Solutions

S1 (a) Period = 4s

Frequency = $\frac{1}{4}$ Hz

$$\omega = \frac{2\pi}{4} = \frac{\pi}{2} \text{ rad/s}$$

$$= 1.5708 \text{ rad/s}$$

(b) Mean position = 0.6

Amplitude = 0.2

It starts at the top, so we will use a cosine function

$$x = 0.6 + 0.2\cos\frac{\pi}{2}t$$

$$\dot{x} = -\frac{\pi}{10}\sin\frac{\pi}{2}t$$

$$\ddot{x} = -\frac{\pi^2}{20}\cos\frac{\pi}{2}t$$

(c)

$$\dot{x} = \pm\omega\sqrt{a^2 - (x-0.6)^2}$$

$$= \pm\frac{\pi}{2}\sqrt{0.2^2 - (x-0.6)^2}$$

$$\ddot{x} = -\omega^2(x-0.6)$$

$$= -\frac{\pi^2}{4}(x-0.6)$$

(d) When t = 2.5

$$x = 0.6 + 0.2\cos\left(\frac{\pi}{2} \times 2.5\right)$$

$$= 0.4586 \text{ m}$$

$$\dot{x} = -\frac{\pi}{10}\sin\left(\frac{\pi}{2} \times 2.5\right)$$

$$= 0.222 \text{ m/s}$$

$$\ddot{x} = -\frac{\pi^2}{20}\cos\left(\frac{\pi}{2} \times 2.5\right)$$

$$= 0.3489 \text{ m/s}^2$$

(e) When x = 0.75

$$\dot{x} = \pm\frac{\pi}{2}\sqrt{0.2^2 - 0.15^2}$$

speed = 0.2078 m/s

$$\ddot{x} = -\frac{\pi^2}{4}(.15)$$

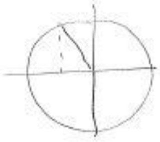
$$= -0.3701 \text{ m/s}^2$$

(f) The weight starts 0.5m above the floor.

When $x = 0.5$ $\quad 0.5 = 0.6 + 0.2\cos\frac{\pi}{2}t$

$-0.1 = 0.2\cos\frac{\pi}{2}t$

$-\frac{1}{2} = \cos\frac{\pi}{2}t$



$\frac{\pi}{2}t = \frac{2\pi}{3}$

$t = \frac{4}{3} = 1.333\,s$

(g) The greatest speed is when $x = 0.6$

Then speed $= \frac{\pi}{2}\sqrt{0.2^2}$

$= 0.1\pi$

$= 0.314\ m/s$

The greatest acceleration occurs at minimum x

~~Then~~ ie when $x = 0.4$

Then $\ddot{x} = \frac{-\pi^2}{4}(-0.2)$

$= 0.05\pi^2$

$= 0.4935\ m/s^2$

(h) When $\ddot{x} = +0.1\,m/s^2$

$\ddot{x} = -\frac{\pi^2}{4}(x-0.6)$

$0.1 = -\frac{\pi^2}{4}(x-0.6)$

$-\frac{0.4}{\pi^2} = x - 0.6$

$x = 0.6 - \frac{0.4}{\pi^2}$

$= 0.5595\ m$

$\dot{x} = \pm\frac{\pi}{2}\sqrt{0.2^2-(x-0.6)^2}$

$= \pm\frac{\pi}{2}\sqrt{0.2^2 - 0.0405^2}$

$= \pm 0.3077\ m/s$

So the speed is $0.3077\ m/s$

S2 (a) $f = 2\ Hz$ $\quad \omega = 4\pi$

When $x = 0.05$ $\quad v = 0.6$

$v = \omega\sqrt{a^2-x^2}$

$0.6 = 4\pi\sqrt{a^2-0.05^2}$

$\sqrt{a^2-0.05^2} = \frac{0.6}{4\pi}$

$a^2 - 0.05^2 = \frac{0.36}{16\pi^2}$

$a^2 = \frac{0.36}{16\pi^2} + 0.05^2$

$= 4.78 \times 10^{-3}$

$a = 0.0691$

∴ The amplitude is 6.91 cm

(b) The greatest speed occurs when $x = 0$

Then $v = \omega a$

$= 4\pi \times .0691$

$= 0.8688\ m/s$

The greatest acceleration occurs when $x = -a$

Then $\ddot{x} = \omega^2 a = (4\pi)^2 \times 0.0691$

$= 109\ m/s^2$

(c) When $x = 3cm$

$v = 4\pi\sqrt{0.0691^2 - 0.03^2}$

$= 0.782\ m/s$

When $x = 3cm$,

$\ddot{x} = (4\pi)^2 \times 0.03$

$= 4.74\ m/s^2$

(d) $x = a\cos\omega t$

$x = 0.0691\cos 4\pi t$

When x is maximum, $t = 0$

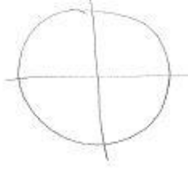
when $x = 0.03$

$0.03 = 0.0691\cos 4\pi t$

$0.4342 = \cos 4\pi t$

$4\pi t = 1.1217$

$t = 0.08926$



∴ It takes 0.08926 s

(e) $V = \omega\sqrt{a^2-x^2}$

$0.3 = 4\pi\sqrt{0.0691^2-x^2}$

$0.02387 = \sqrt{0.0691^2-x^2}$

$0.0005699 = 0.0047748 - x^2$

$x^2 = 0.0042049$

$x = 0.0648$

$= 6.48\text{cm}$

S3 $\ddot{x} = -4x$

$\therefore \omega^2 = 4, \quad \omega = 2$

$\therefore$ Period $= \frac{2\pi}{\omega} = \pi$

The greatest speed is when $x = 0$

$\therefore 4 = \pi\sqrt{a^2-0}$

$\frac{4}{\pi} = a$

$\therefore$ The amplitude is $4/\pi = 1.273$

S4 $\dot{x} = 8\sqrt{9-x^2}$

$\omega = 8, \quad a = 3$

$\ddot{x} = -64x$

Period $= \frac{2\pi}{\omega} = \frac{\pi}{4}$

Amplitude $= 3$

Greatest speed is $\dot{x}$ when $x = 0$

$= 8\sqrt{9}$

$= 24$

S5 (a) $x = 4\cos 3t$

$\dot{x} = -12\sin 3t$

$\ddot{x} = -36\cos 3t$

(b) Amplitude $= 4$

$\omega = 3$

Period $= \frac{2\pi}{\omega} = \frac{2\pi}{3}$

(c) The greatest speed is when $\sin 3t = 1$

$\therefore$ Greatest speed is 12

The greatest acceleration is when $\cos 3t = 1$

ie 36

(d) $V = \pm\omega\sqrt{a^2-x^2}$

$= \pm 3\sqrt{16-x^2}$

$\ddot{x} = -\omega^2 x$

$= -9x$

S6 (a) When $x = 1.2 \quad \ddot{x} = 2.4 \quad \dot{x} = 2$

$\ddot{x} = \omega^2 x$

$2.4 = \omega^2 \times 1.2$

$2 = \omega^2$

$\omega = \sqrt{2}$

Period $= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

$\dot{x} = \omega\sqrt{a^2-x^2}$

$2 = \sqrt{2}\sqrt{a^2-1.2^2}$

$4 = 2(a^2-1.44)$

$2 = a^2 - 1.44$

$3.44 = a^2$

$a = 1.855 \text{ m/s}^2$

(b) ~~wh~~ When $x = 0$

$V = \omega\sqrt{a^2}$

$= \sqrt{2} \times 1.855$

$= 2.623 \text{ m/s}$

(c) When $x = 1.5$

$V = \sqrt{2}\sqrt{1.855^2 - 1.5^2}$

$= 1.543 \text{ m/s}$

$\ddot{x} = \omega^2 x$

$= 2 \times 1.5$

$= 3 \text{ m/s}^2$