Variable Acceleration

Air resistance = kv – Finding v(t)

1. A 100 g air hockey puck is hit so that it slides across a flat surface with an initial speed of 4 m/s. The resistance to movement in Newtons is 0.02 times the velocity in m/s. Find:
   a. The relation between the speed and the time since being hit
   b. The speed 3 seconds after being hit
   c. The relation between distance travelled and time since being hit
   d. How far it moves in the first 5 seconds
   e. Its speed when it has moved 3 m
   f. How far it moves before it stops.

2. A ball is dropped from the top of a cliff. Gravity provides an acceleration of 9.8 m/s^2 downwards and air resistance provides an acceleration of $0.4v$ m/s^2 upwards, where v is the speed in m/s. Find:
   a. The relation between the speed and the time since it fell
   b. The speed 3 seconds after it fell
   c. The relation between distance travelled and time since it fell
   d. How far it moves in the first 5 seconds
   e. How long it takes to fall 60 m
   f. Its speed when it has fallen 30 m
   g. Its terminal velocity.

Air resistance = kv² – Finding v(x)

3. A 100 g air hockey puck is hit so that it slides across a flat surface with an initial speed of 4 m/s. The resistance to movement in Newtons is $0.002v^2$, where v is the velocity in m/s. Find:
   a. The relation between speed and distance travelled
   b. The speed when it has travelled 2 m
   c. The distance travelled when its speed is halved
   d. The total distance travelled

4. A ball is dropped from a hot air balloon. Gravity provides an acceleration of 9.8 m/s^2 downwards and air resistance provides an acceleration of $0.05v^2$ m/s^2 upwards, where v is the speed in m/s. Find:
   a. The relation between the speed and distance fallen
   b. The speed when it has fallen 4 m
   c. The distance fallen when its velocity reaches 12 m/s
   d. The terminal velocity

5. A ball is thrown upwards from the ground at 8 m/s. Gravity provides an acceleration of 9.8 m/s^2 downwards and air resistance provides an acceleration of $0.1v^2$ m/s^2 opposing the motion, where v is the speed in m/s. Find:
   a. The relation between the speed and height above the ground on the way up
   b. The speed 2 m above the ground on the way up
   c. The maximum height reached

6. A bullet is fired vertically upwards with an initial speed of u m/s against an air resistance equal to kv^2, where v is the velocity in m/s and k is a constant. Show that the bullet rises a distance given by

$$\frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$$

Acceleration given in terms of x

In the following questions, take the radius of the Earth to be 6400 km and the gravity at the surface to be 9.8 m/s2. Ignore air resistance.

7. A rocket is launched vertically from the Earth's surface. It reaches 4000 m/s before its fuel runs out at an altitude of 200 km. Find:
    a. the relation between velocity and height after the fuel runs out
    b. The velocity at a height of 500 km
    c. the height (above 200 km) when the velocity is 2000 m/s
    d. the greatest height reached

8. A missile is launched from the Earth's surface and moves vertically without further propulsion.
    a. Find the launch speed required for it to reach a maximum height of 1000 km
    b. Find the escape velocity, i.e. the launch speed required for it to just escape the Earth's gravity (in other words for it to reach a maximum height of infinity)

9. A proton is fired towards a positively charged target. The initial distance is 2 m and the initial velocity is 100 m/s. The acceleration on the proton resulting from the electrostatic repulsion is given by $500/x^2$ m/s^2, where x is the distance from the target. Gravity can be ignored.
    a. Find the relation between velocity and x.
    b. Find the velocity when the proton is 20 cm from the target.
    c. Find the velocity when the proton is 5 cm from the target. Discuss your result.
    d. How close does the proton get to the target?
    e. What will be the velocity of the proton when it is once again 60 cm from the target?

10. A 200 g charged ball is held 2 m above a charged plate then dropped. Assuming the electrostatic repulsion of the ball is $^2/_h$ N, where h is the height of the ball above the plate, find:
    a. the relation between velocity of the ball and height
    b. the velocity when $h = 1$ m
    c. the closest the ball gets to the plate
    d. the maximum speed of the ball on the way down.

Answers: 1. (a) $v = 4e^{-0.2t}$ (b) 2.195 m/s (c) $s = 20(1-e^{-0.2t})$ (d) 12.64 m (e) 3.40 m/s (f) 20 m 2. (a) $v = 24.5(1-e^{-0.4t})$ (b) 17.12 m/s (c) $s = 24.5t + 61.25(e^{-0.4t}-1)$ (d) 69.54 m (e) 4.54 s (f) 17.00 m/s (g) 24.5 m/s 3. (a) $v = 4e^{-0.02x}$ (b) 3.843 m/s (c) 34.66 m (d) ∞ 4. (a) $v^2 = 196(1-e^{-0.1x})$ (b) 8.04 m/s (c) 13.27 m (d) 14 m/s 5. (a) $v^2 = 162e^{-0.2x} - 98$ (b) 3.25 m/s (c) 2.51 m 7. (a) $v^2 = 802.8\times10^{12} \div (6.4\times10^6 + h) - 105.6\times10^6$ (b) 3278 m/s (c) 925 km (d) 1202 km 8. (a) 4062 m/s (b) 11 200 m/s 9. (a) $v^2 = 10\,500 - 1000/x$ (b) 74.16m/s (c) $97.5i$ This shows that the proton doesn't get this close (d) 9.52 cm (e) 100 m/s 10. (a) $v^2 = -19.6h + 20 \ln h + 25.34$ (b) 2.40 m/s (c) 42.9 cm (d) 2.40 m/s

Solutions

Q1.(a) $a = 0.2v \text{ m/s}^2$

$$\frac{dv}{dt} = -0.2v$$

$$\int \frac{dv}{0.2v} = -\int dt$$

$$5\ln(0.2v) = -t + c$$

$$\ln 0.2v = -0.2t + d$$

$$0.2v = e^{-0.2t+d}$$

$$v = Ae^{-0.2t}$$

When $t = 0$, $v = 4$

$$4 = A$$

$$\therefore v = 4e^{-0.2t}$$

(b) When $t = 3$, $v = 4e^{-0.6} = 2.195 \text{ m/s}$

(c) $$\frac{ds}{dt} = 4e^{-0.2t}$$

$$s = -\frac{4}{0.2}e^{-0.2t} + c$$

$$s = -20e^{-0.2t} + c$$

When $t = 0$, $s = 0$

$$0 = -20 + c$$

① (f) When $t = \infty$

$$s = 20(1 - 0)$$

$$= 20 \text{ m}$$

② (b) When $t = 3$

$$v = 24.5(1 - e^{-1.2})$$

$$= 17.12 \text{ m/s}$$

(c) $$\frac{ds}{dt} = 24.5(1 - e^{-0.4t})$$

$$= 24.5 - 24.5e^{-0.4t}$$

$$s = 24.5t + \frac{24.5}{0.4}e^{-0.4t} + c$$

$$s = 24.5t + 61.25e^{-0.4t} + c$$

When $t = 0$, $s = 0$

$$\therefore 0 = + 61.25 + c$$

$$c = -61.25$$

$$\therefore s = -61.25 + 24.5t + 61.25e^{-0.4t}$$

$$s = 24.5t + 61.25(e^{-0.4t} - 1)$$

Q3. (a) $a = -0.02v^2$

$$v\frac{dv}{dx} = -0.02v^2$$

$$\int \frac{v\,dv}{v^2} = -0.02\int dx$$

$$\int \frac{dv}{v} = -0.02\int dx$$

$$\ln v = -0.02x + c$$

When $x = 0$, $v = 4$

$$\ln 4 = c$$

$$\therefore \ln v = -0.02x + \ln 4$$

$$\ln v - \ln 4 = -0.02x$$

$$\ln \frac{v}{4} = -0.02x$$

$$\frac{v}{4} = e^{-0.02x}$$

$$v = 4e^{-0.02x}$$

(b) When $x = 2$, $v = 4e^{-0.04}$

$$v = 3.843 \text{ m/s}$$

Q4. (a) Let down be positive

$$a = 9.8 - 0.05v^2$$

$$v\frac{dv}{dx} = 9.8 - 0.05v^2$$

$$\int \frac{v}{9.8-0.05v^2}\,dv = \int dx$$

$$\frac{\ln(9.8-0.05v^2)}{-0.1} = x + c$$

$$-10\ln(9.8-0.05v^2) = x + c$$

When $x = 0$ $v = 0$

$$\therefore -10\ln 9.8 = c$$

$$-10\ln(9.8-0.05v^2) = -10\ln 9.8 + x$$

$$10\ln 9.8 - 10\ln(9.8-0.05v^2) = x$$

$$10\ln\left(\frac{9.8}{9.8-0.05v^2}\right) = x$$

$$\ln\frac{9.8}{9.8-0.05v^2} = 0.1x$$

$$\frac{9.8}{9.8-0.05v^2} = e^{0.1x}$$

(4) (d) $V^2 = 196(1 - e^{-x/10})$

At terminal velocity $x = \infty$

$$\therefore v^2 = 196$$

$$v = 14 \text{ m/s}$$

(5) (b) When $x = 2$

$$v^2 = 162e^{-0.4} - 98$$

$$= 10.59$$

$$v = 3.25 \text{ m/s}$$

(c) The maximum height is when $V = 0$

$$0 = 162e^{-0.2x} - 98$$

$$162e^{-0.2x} = 98$$

$$e^{-0.2x} = \frac{98}{162}$$

$$-0.2x = \ln\left(\frac{98}{162}\right)$$

$$x = -5\ln\left(\frac{98}{162}\right)$$

$$= 2.51$$

$\therefore$ The maximum height is 2.51 m

Q7. (a) Taking up as positive and x as the distance from the centre of the Earth

$$a = -\frac{g(6.4\times10^6)^2}{x^2}$$

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -9.8\times6.4^2\times10^{12}\times x^{-2}$$

$$\tfrac{1}{2}v^2 = 9.8\times6.4^2\times10^{12}\times x^{-1} + c$$

$$v^2 = 802.8\times10^{12}x^{-1} + d$$

When $x = 6.6\times10^6$ $v = 4000$

$$\therefore 16\times10^6 = \frac{802.8\times10^{12}}{6.6\times10^6} + d$$

$$d = 16\times10^6 - \frac{802.8\times10^{12}}{6.6\times10^6}$$

$$= -105.6\times10^6$$

$$\therefore v^2 = 802.8\times10^{12}x^{-1} - 105.6\times10^6$$

$$x = 6.4\times10^6 + h$$

$$\therefore v^2 = \frac{802.8\times10^{12}}{6.4\times10^6 + h} - 105.6\times10^6$$

(7)(d) At the greatest height $v = 0$

$$0 = \frac{802.8 \times 10^{12}}{6.4\times10^6 + h} - 105.6\times10^6$$

$$105.6\times10^6 \times (6.4\times10^6 + h) = 802.8\times10^{12}$$

$$6.4\times10^6 + h = \frac{802.8\times10^{12}}{105.6\times10^6}$$

$$h = \frac{802.8\times10^{12}}{105.6\times10^6} - 6.4\times10^6$$

$$= \underline{1202 \text{ km}}$$

(8) $v^2 = 125\times10^6 - 108.5\times10^6$

$= 16.5\times10^6$

$v = 4.062\times10^3$

∴ The launch speed required is 4062 m/s

(b) To escape, $v = 0$ when $x = \infty$

$$v^2 = \frac{802.8\times10^{12}}{x} + d$$

$0 = 0 + d$

$$\therefore\ v^2 = \frac{802.8\times10^{12}}{x}$$

Launch is at $x = 6.4\times10^{12}$

Then $v^2 = \dfrac{802.8\times10^6}{6.4}$

$= 125\times10^6$

$v = 11200$

∴ The escape velocity is 11200 m/s

(9)(c) When $x = 0.05$

$$v^2 = 10500 - \frac{1000}{0.05}$$

$= -9500$

$v = 97.5i$

This shows that the proton doesn't get this close to the target

(d) When the proton is at the minimum distance

$v = 0$

$$\therefore\ 0 = 10500 - \frac{1000}{x}$$

$$\frac{1000}{x} = 10500$$

$$x = \frac{1000}{10500}$$

$= 0.0952$ m

$= \underline{9.52 \text{ cm}}$

(e) Because energy is not lost, it will again be moving at 100 m/s, this time away from the target

(10)(c) At the closest point $v = 0$

$\therefore\ 0 = -19.6h + 20\ln h + 25.34$

$19.6h - 20\ln h = 25.34$

Solving graphically gives

$h = 0.429$ m

$= \underline{42.9 \text{ cm}}$

(d) The maximum speed is where $a = 0$

ie $0 = -9.8 + \frac{2}{h} \div 0.2$

$$0 = -9.8 + \frac{10}{h}$$

$$9.8 = \frac{10}{h}$$

$$h = \frac{10}{9.8}$$

$= 1.02$ m

At that height $v^2 = -19.6\times1.02 + 20\ln 1.02 + 25.34$

$= 5.744$

$v = 2.40$

∴ The maximum speed is 2.40 m/s