Applications of $\frac{dy}{dx} = f(y)$

1. The decay of a radioactive isotope is modeled by $\frac{dA}{dt} = kA$, where A is the amount present in milligrams and t is the time in minutes. The isotope has a half life of 40 min.
   a. Find the value of k
   b. Hence find the fraction remaining after 110 min.

2. The same isotope as in Question 1 is produced at a rate of 12 mg/min. How much is present 60 min after production begins?

3. An isotope is produced at 30 mg/min and has a half-life of 8 min.
   a. Find an expression for the amount present t min after production begins
   b. Hence find the amount present 21 min after production begins
   c. How long will it take for 0.2 gram to be present?
   d. What is the maximum amount that will ever be present?

4. An isotope is produced at 200 mg/min and has a half-life of 15 min. Find the amount present 35 min after production begins.

5. A boat is moving across a lake at 5 m/s. It switches its motor off. After that, the decelerat-ion (in m/s^2) due to drag from the water and air is equal to 0.04 times the speed (in m/s).
   a. Find the relation between speed and time from the moment the motor is turned off.
   b. Find the relation between distance travelled and time.
   c. Find the distance travelled after 20 s.
   d. How far will the boat travel before it stops?

6. A 0.2 kg ball is dropped from the top of a building. It experiences a gravitational force of 1.96 N and an air resistance of $0.1v$ N, where v is the speed in m/s.
   a. Find the relation between speed and time.
   b. Find the relation between distance fallen and time.
   c. How long will it take to hit the ground 60 m below?

7. The rate at which dirt accumulates on a kitchen bench is inversely proportional to the thickness already present. If it takes 20 days for the dirt to get 4 mm thick, how long will it take for a further 6 mm to accumulate?

8. As a space capsule falls towards the Earth, the rate of increase of kinetic energy with respect to distance from the centre of the Earth is inversely proportional to the square of the distance. Assume that the kinetic energy is 200 MJ when the distance is 10 000 km and that, at that distance, it increases at 5 MJ/km, what will be the kinetic energy when it enters the atmosphere 6600 km from the centre of the Earth?

9. Atmospheric pressure at a point is equal to the weight of air above a unit area at that point. The density of air is proportional to its pressure. At sea level the density is 1.25 kg/m^3 and the pressure is 101 300 N/m^2.
   a. Find the relation between pressure and height above sea level.
   b. Find the pressure at 4 000 m.
   c. Find the height at which the pressure is half that at sea level.

Answers

1. a. −0.0173 b. 14.9% 2. 448 mg 3. a. $A = \frac{30-30e^{-0.0866t}}{0.0866}$ b. 290 mg c. 9.94 min d. 346 mg 4. 3470 mg 5. a. $v = 5e^{-0.04t}$ b. $s = 125\,(1 - e^{-0.04t})$ c. 68.8 m d. 125 m 6. a. $v = 19.6\,(1 - e^{-0.5t})$ b. $s = 19.6t + 39.2(e^{-0.5t} - 1)$ c. 4.88 s 7. 105 d 8. 25 957 MJ 9. a. $P = 101300e^{-0.000121h}$ b. 62 457 N/m^2 c. 5733 m

Solutions

Applications of $\frac{dy}{dx} = f(y)$

(1) (a) $\frac{dA}{dt} = kA$

$\int \frac{dA}{A} = k\int dt$

$\ln A = kt + c$

$A = e^{kt+c}$

$A = A_0 e^{kt}$

Half life is 40 min

$\therefore \frac{1}{2} = e^{40k}$

$\ln \frac{1}{2} = 40k$

$\ln 2 = -40k$

$k = -\frac{\ln 2}{40}$

$= -0.0173$

(b) When $t = 110$

$\frac{A}{A_0} = e^{-0.0173 \times 110}$

Fraction remaining $= 0.149$

$= 14.9\%$

(2) $\frac{dA}{dt} = 12 + kA$

$\frac{dA}{dt} = 12 - 0.0173A$

$\int \frac{dA}{12 - 0.0173A} = \int dt$

$\frac{-\ln(12 - 0.0173A)}{0.0173} = t + c$

When $t = 0$, $A = 0$

$\therefore -\frac{\ln 12}{0.173} = c$

$\frac{-\ln(12 - 0.0173A)}{0.0173} = t - \frac{\ln 12}{0.0173}$

$-\ln(12 - 0.0173A) = 0.0173t - \ln 12$

$12 - 0.0173A = e^{\ln 12 - 0.0173t}$

$A = \frac{12 - e^{\ln 12 - 0.0173t}}{0.0173}$

When $t = 60$ $A = \frac{12 - e^{\ln 12 - 0.0173 \times 60}}{0.0173}$

$= 448$ mg

(3) (a) Assuming it is not being produced

$\frac{dA}{dt} = kA$

$\int \frac{dA}{A} = \int k \, dt$

$\ln A = kt + c$

$A = e^{kt+c}$

$A = A_0 e^{kt}$

When $t = 8$ $\quad \frac{1}{2} = e^{8k}$

$\ln \frac{1}{2} = 8k$

$k = \frac{\ln \frac{1}{2}}{8}$

$= -0.0866$

$\therefore \frac{dA}{dt} = 30 - 0.0866A$

$\int \frac{dA}{30 - 0.0866A} = \int dt$

$-\frac{\ln(30 - 0.0866A)}{0.0866} = t + c$

$\ln(30 - 0.0866A) = -0.0866t + d$

When $t = 0$ $A = 0$

$\therefore \ln 30 = d$

$\therefore \ln(30 - 0.0866A) = \ln 30 - 0.0866t$

$30 - 0.0866A = 30e^{-0.0866t}$

$0.0866A = 30(1 - e^{-0.0866t})$

$A = 346(1 - e^{-0.0866t})$

(b) When $t = 21$, $A = 346(1 - e^{-0.0866 \times 21})$

$= 290$ mg

(c) $0.2g = 200$ mg

$200 = 346(1 - e^{-0.0866t})$

$0.5780 = 1 - e^{-0.0866t}$

$0.4220 = e^{-0.0866t}$

$-\ln 0.4220 = 0.0866t$

$t = 9.96$ min

(d) The maximum A will occur when $t \to \infty$

Then $A = 346(1 - 0)$

$= 346$ mg

(4) $\frac{dA}{dt} = kA$ if none is produced

$\int \frac{dA}{A} = \int kt$

$\ln A = kt + c$

$A = e^{kt+c}$

$A = A_0 e^{kt}$

When $A/A_0 = \frac{1}{2}$ $\quad \frac{1}{2} = e^{kt}$

$\ln \frac{1}{2} = kt$

$t = 15$

$\therefore \ln \frac{1}{2} = 15k$

$k = \frac{\ln \frac{1}{2}}{15}$

$= -0.0462$

with production

$\frac{dA}{dt} = 200 - 0.0462A$

$\int \frac{dA}{200 - 0.0462A} = \int dt$

$-\frac{\ln(200 - 0.0462A)}{0.0462} = t + C$

When $t = 0$, $A = 0$

$\therefore -\frac{\ln 200}{0.0462} = c$

$\therefore -\frac{\ln(200 - 0.0462A)}{0.0462} = t - \frac{\ln 200}{0.0462}$

$-\ln(200 - 0.0462A) = 0.0462t - \ln 200$

$-\ln(200 - 0.0462A) + \ln 200 = 0.0462t$

$\ln\left(\frac{200 - 0.0462A}{200}\right) = -0.0462t$

When $t = 35$

$\ln\left(\frac{200 - 0.0462A}{200}\right) = -0.0462 \times 35$

$= -1.617$

$\frac{200 - 0.0462A}{200} = e^{-1.617}$

$200 - 0.0462A = 200e^{-1.617}$

$-0.0462A = 200e^{-1.617} - 200$

$A = \frac{200(1 - e^{-1.617})}{0.0462}$

$= 3470$ mg

(5) (a) Let v be the speed in m/s at time t (seconds) after the motor is turned off.

$\frac{dv}{dt} = -0.04v$

$\frac{dv}{v} = -0.04\,dt$

$\ln v = -0.04t + c$

$v = e^{-0.04t + c}$

$v = Ae^{-0.04t}$

When $t = 0$ $v = 5$

$5 = A$

$\therefore v = 5e^{-0.04t}$

(b) Let x be the distance travelled (in m) at time t

Then $\frac{dx}{dt} = 5e^{-0.04t}$

$x = \frac{5e^{-0.04t}}{-0.04} + c$

When $t = 0$ $x = 0$

$\therefore 0 = \frac{5}{-0.04} + c$

$c = 125$

$\therefore x = -\frac{5e^{-0.04t}}{0.04} + 125$

$= -125e^{-0.04t} + 125$

$= 125(1 - e^{-0.04t})$

(c) When $t = 20$

$x = 125(1 - e^{-0.8})$

$= 68.8$ m

(d) It will stop when $t \to \infty$

Then $x = 125$

$\therefore$ It will travel 125 m

(6) (a) Take downwards as positive

$$\frac{dv}{dt} = 9.8 - 0.5v$$

$$\int \frac{dv}{9.8-0.5v} = \int dt$$

$$-2\ln(9.8-0.5v) = t + c$$

When $t=0$ $v=0$

$$-2\ln 9.8 = c$$

$$\therefore -2\ln(9.8-0.5v) = t - 2\ln 9.8$$

$$2(\ln 9.8 - \ln 9.8-0.5v) = t$$

$$\ln\left(\frac{9.8-0.5v}{9.8}\right) = -\frac{t}{2}$$

$$\frac{9.8-0.5v}{9.8} = e^{-t/2}$$

$$9.8-0.5v = 9.8e^{-t/2}$$

$$0.5v = 9.8(1-e^{-t/2})$$

$$v = 19.6(1-e^{-t/2})$$

(b) $$\frac{ds}{dt} = 19.6(1-e^{-t/2})$$

$$s = 19.6(t + 2e^{-t/2}) + c$$

When $t=0$ $s=0$

$$\therefore 0 = 19.6(2) + c$$

$$c = -39.2$$

$$\therefore s = 19.6(t+2e^{-t/2}) - 39.2$$

$$= 19.6t + 39.2e^{-t/2} - 39.2$$

$$= 19.6t + 39.2(e^{-t/2}-1)$$

(c) It hits the ground when

$$60 = 19.6t + 39.2(e^{-t/2}-1)$$

Solving graphically gives $t = 4.89s$

$\therefore$ It takes 4.89s to hit the ground

(7) Let the thickness present at time t be T where t is in days and T in mm

$$\frac{dT}{dt} = \frac{k}{T}$$

$$\int T\,dT = k\int dt$$

$$\frac{T^2}{2} = kt + c$$

When $t=0$ $T=0$

So $0 = 0 + c$

$\therefore c = 0$

$\therefore \frac{T^2}{2} = kt$ ie $T^2 = 2kt$

When $t = 20$, $T = 4$

$$\therefore \frac{4^2}{2} = k \times 20$$

$$k = \frac{8}{20} = 0.4$$

$$\therefore T^2 = 0.8t$$

$$T = \sqrt{0.8t}$$

When $T = 10$ $\quad 10 = \sqrt{0.8t}$

$$100 = 0.8t$$

$$t = 125$$

$\therefore$ It takes a further 105 days

(8) Let the kinetic Energy (in MJ) be E at distance x (in km) from the centre of the Earth

$$\frac{dE}{dx} = -\frac{k}{x^2}$$

when $x = 10000$ $\frac{dE}{dx} = -5$

$$-5 = \frac{-k}{10^8}$$

$$k = 5\times10^8$$

$$\therefore \frac{dE}{dx} = \frac{-5\times10^8}{x^2}$$

$$\frac{dE}{dx} = -5\times10^8 x^{-2}$$

$$E = 5\times10^8 x^{-1} + c$$

When $x = 10000$ $E = 200$

$$\therefore 200 = 5\times10^4 + c$$

$$c = 200 - 50000$$

$$= -49800$$

$$\therefore E = 5\times10^8 x^{-1} - 49800$$

When $x = 6600$

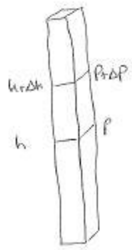
$$E = \frac{5\times10^8}{6600} - 49800 = 25957\text{ MJ}$$

(9) (a) Consider a $1m^2$ column of air from sea-level to the top of the atmosphere

Consider a section of this column from height h to height $h+\Delta h$

Let the pressure at the bottom of the section be P and at the top $P+\Delta P$

ΔP = the weight of air in that section



The density of this air is

$$1.25 \times \frac{P}{101300} = 0.00001234P$$

$\therefore$ The mass is $0.00001234 P \Delta h$

The weight is $9.8 \times 0.00001234 P\Delta h$

$$= 0.0001209 P \Delta h$$

$$\therefore \quad \Delta P = -0.0001209 P \Delta h$$

$$dP = -0.0001209 P \, dh$$

$$\int \frac{dP}{P} = -0.0001209 \int dh$$

$$\ln P = -0.0001209h + c$$

When $h = 0$ $P = 101300$

$\therefore$ $\ln 101300 = c$

$$\therefore \quad \ln P = -0.0001209h + \ln 101300$$

$$\ln P - \ln 101300 = -0.0001209h$$

$$\ln \frac{P}{101300} = -0.0001209h$$

$$P = 101300e^{-0.0001209h}$$

(b) When $h = 4000$

$$P = 101300e^{-0.0001209 \times 4000}$$

$$= 62457 \text{ N/m}^2$$

(c) When $P = \frac{101300}{2}$

$$\frac{1}{2} = e^{-0.0001209h}$$

$$\ln 2 = 0.0001209h$$

$$h = \frac{\ln 2}{0.0001209}$$

$$= 5733 \text{ m}$$