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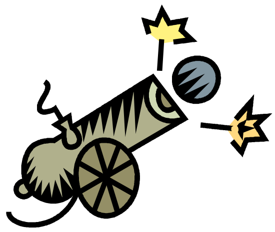
1. A cannon ball is fired out to sea from the top of a vertical cliff 60 m above sea level. It is fired at 45° above the horizontal with a speed u.



   (a) Assuming the cannon to be at the origin, g to be 9.8 m/s^2 and air resistance to be negligible, find vector expressions for the acceleration, velocity and position of the cannonball t seconds after firing.

   (b) Hence find the value of u, given that the cannon ball hit a canoeist 175 m from the base of the cliff.

2. Solve $\dfrac{dN}{dt} = 4N + 3$ if $N = 20$ when $t = 0$

3. A radioactive isotope is produced at 10 mg/min. It has a half-life of 90 min. Find the amount present 120 minutes after production begins.

4. Greg is standing on a horizontal rotating platform. He is 4 m from the axis of rotation. If the coefficient of friction between Greg and the platform is 0.3, at what angular velocity will the platform have to spin to make Greg fall off? [Assume g = 9.8 m/s^2.]

5. A particle is moving in a vertical line such that its height, x, above a fixed point at any time t is given by $x = a \sin \omega t$, where a and ω are constants. If the velocity at any time is v, show that the relation between v and x is $v = \pm\omega\sqrt{a^2 - x^2}$.

6. A rocket is launched from the surface of the Earth. The fuel runs out at a height of 200 km. By then the rocket is moving vertically upwards at 5000 m/s. Find the maximum height the rocket will reach. [Take the radius of the Earth to be 6400 km and gravity at the surface to be 9.8 m/s^2. Assume that gravitational acceleration is inversely proportional to the square of the distance from the centre of the Earth.]

7. A cork is projected vertically upwards from a bottle with an initial speed of u m/s against an air resistance equal to kv^2, where v is the velocity in m/s at any instant and k is a constant. Show that the cork rises a height given by

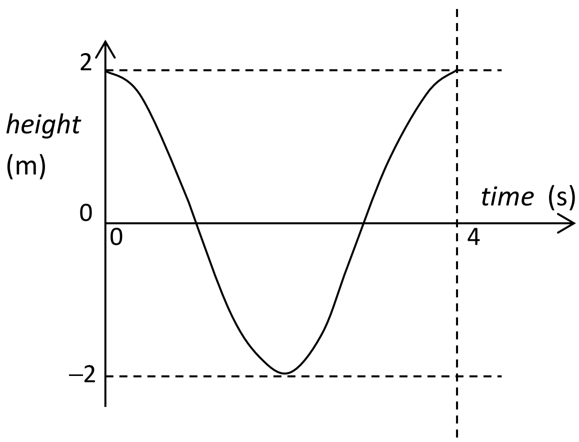
$$\frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$$

1. If $xy\frac{dy}{dx}-(1+x^2)(1-y^2)=0$ and $y = 0$ when $x = 1$, find y when $x = 2$.

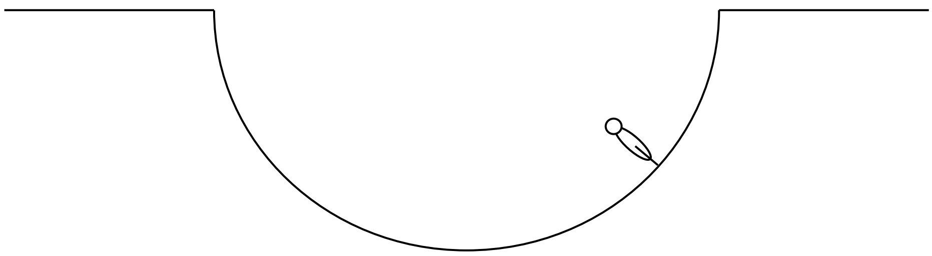
2. As ice forms on a lake, the thickening layer of ice provides increasing insulation for the water underneath. Thus, for a given air temperature, the rate of increase of ice thickness is inversely proportional to the thickness of ice already present. It takes 4 hours for the ice to reach a thickness of 1 cm. Produce an algebraic model for the time taken to reach any thickness and use it to find how long it would take to reach 2.5 cm?
Justify the reasonableness of your result.

3. A glass tube contains a fixed magnet at its lower end. A movable cylindrical magnet with mass 200 g is held at the upper end of the tube 2 m above the fixed magnet and oriented so as to be repelled by the fixed magnet. The movable magnet is acted upon by gravity (g = 9.8 m/s²) and a repulsive force from the fixed magnet given by $F=\frac{1}{2x^3}$, where F is the repulsion in Newtons and x is the distance from the fixed magnet in metres. The movable magnet is then dropped. Find how close the movable magnet gets to the fixed magnet before moving away again.
What assumptions are necessary in obtaining your result?

4. The graph below shows approximately the displacement of a 250 g German sausage moving up and down on the end of a long light spring.
(a) Select an appropriate model for the motion and use it to find the greatest acceleration.
(b) If g = 9.8 m/s², find the greatest tension in the spring.
(c) Generalise this result for any sausage mass.

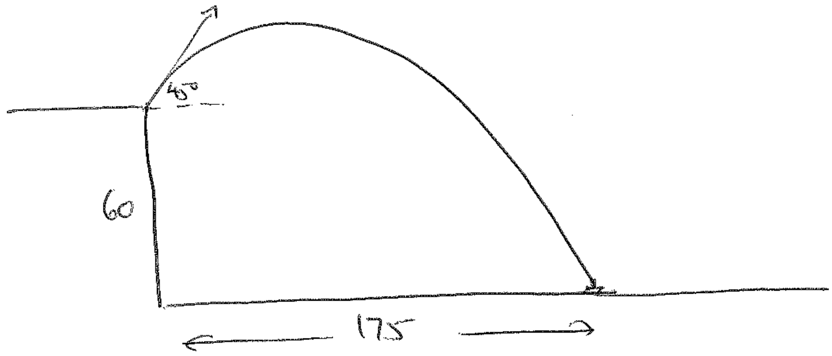


5. A cycle track consists of a hemispherical bowl 40 m in diameter. A cyclist rides around in circles inside the bowl at 54 km/h, keeping the same height above the bottom of the bowl. If she is perpendicular to the surface on which she is riding (i.e. there is no friction between her bike and the surface), how long does it take her to complete one lap?



Year 12 Maths C Term 3 Paper A 2012 - Solutions

(1)



a) Let $\underset{\sim}{i}$ be the unit vector to the right
Let $\underset{\sim}{j}$ be the unit vector upwards
Let $\underset{\sim}{a}$ be the acceleration, $\underset{\sim}{v}$ the velocity, $\underset{\sim}{s}$ the displacement at time t (units metres and seconds)

$$\underset{\sim}{a} = 0\underset{\sim}{i} - 9.8\underset{\sim}{j}$$

$$\underset{\sim}{v} = \int (0\underset{\sim}{i} - 9.8\underset{\sim}{j})\,dt$$

$$= c\underset{\sim}{i} - 9.8t\underset{\sim}{j} + d\underset{\sim}{j}$$

When $t=0$ $\quad c = \frac{u}{\sqrt{2}} \quad d = \frac{u}{\sqrt{2}}$

$$\therefore \underset{\sim}{v} = \frac{u}{\sqrt{2}}\underset{\sim}{i} + \frac{u}{\sqrt{2}}\underset{\sim}{j} - 9.8t\underset{\sim}{j}$$

i contd

$$\underset{\sim}{s} = \int\left(\frac{u}{\sqrt{2}}\underset{\sim}{i} + \frac{u}{\sqrt{2}}\underset{\sim}{j} - 9.8t\underset{\sim}{j}\right)dt$$

$$= \frac{ut}{\sqrt{2}}\underset{\sim}{i} + e\underset{\sim}{i} + \frac{ut}{\sqrt{2}}\underset{\sim}{j} - 4.9t^2\underset{\sim}{j} + f\underset{\sim}{j}$$

When $t = 0$ $\underset{\sim}{s} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

$\therefore \quad e = f = 0$

$$\therefore \quad \underset{\sim}{s} = \frac{ut}{\sqrt{2}}\underset{\sim}{i} + \frac{ut}{\sqrt{2}}\underset{\sim}{j} - 4.9t^2\underset{\sim}{j}$$

b) When the horizontal component of $\underset{\sim}{s}$ is 175, the vertical component is -60

ie When $\frac{ut}{\sqrt{2}} = 175$ $\quad \frac{ut}{\sqrt{2}} - 4.9t^2 = -60$

when $t = \frac{175\sqrt{2}}{u}$ $\quad \frac{ut}{\sqrt{2}} - 4.9t^2 = -60$

$$\text{So} \quad \frac{u \times 175\sqrt{2}}{u\sqrt{2}} - 4.9 \times \frac{175^2 \times 2}{u^2} = -60$$

$$175 - \frac{4.9 \times 175^2 \times 2}{u^2} = -60$$

$$\frac{4.9 \times 175^2 \times 2}{u^2} = 235$$

1 contd

$$u^2 = \frac{4.9 \times 175^2 \times 2}{235}$$

$$= 1277.13$$

$$u = 35.7 \text{ m/s}$$

2. $\frac{dN}{dt} = 4N + 3$

$$\frac{dN}{4N+3} = dt$$

$$\int \frac{dN}{4N+3} = \int dt$$

$$\frac{1}{4}\ln(4N+3) = t + c$$

$$\ln(4N+3) = 4t + 4c$$

$$4N + 3 = e^{4t}e^{4c}$$

$$N = Ae^{4t} - \frac{3}{4} \; where \; A = \frac{e^{4c}}{4}$$

at $\mathrm{t} = 0$ $\mathrm{N} = 20$

$$20 = \mathrm{A} - \frac{3}{4}$$

$$\mathrm{A} = 20\frac{3}{4} \text{ or } \frac{83}{4}$$

$$\therefore \mathrm{N} = \frac{83}{4}\mathrm{e}^{4\mathrm{t}} - \frac{3}{4}$$

3. Assume not being produced and determine k for half life

$$\frac{dA}{dt} = -kA$$

$$\int \frac{dA}{A} = -\int kdt$$

$$lnA = -kt + c$$

$$A = e^{-kt}e^{c}$$

$$A = A_0e^{-kt}$$

$$for\ half\ life\ let\ A = 0.5\ and\ A_0 = 1$$

$$0.5 = e^{-90k}$$

$$k = \frac{\ln(0.5)}{-90}$$

$$k = 0.00770$$

$$Now\ consider\ the\ porduction, so$$

$$\frac{dA}{dt} = 10 - kA$$

$$\int \frac{dA}{10 - kA} = \int dt$$

$$-\frac{1}{k}\ln(10 - kA) = t + c$$

$$\ln(10 - kA) = -kt - kc$$

$$10 - kA = e^{-kt}e^{-kc}$$

$$A = \frac{10 - A_0e^{-kt}}{k}$$

$$\text{when t} = 0\ \text{A} = 0$$

$$\therefore A_0 = 10$$

$$\text{A} = \frac{10 - 10\text{e}^{-0.00770\text{t}}}{0.00770}$$

$$when\ t = 120 \quad A = 783mg$$

$$\therefore 783mg\ present\ after\ 2\ hours$$

(4) Let the angular velocity at which Greg falls off be ω

$$m\omega^2 r = 0.3\,mg$$

$$\omega^2 \times 4 = 0.3 \times 9.8$$

$$\omega^2 = \frac{0.3 \times 9.8}{4}$$

$$= 0.735$$

$$\omega = 0.857 \text{ radians/s}$$

(5) $x = a\sin\omega t$

$$v = \dot{x} = \omega a\cos\omega t$$

$$v^2 = \omega^2(a\cos\omega t)^2$$

$$v^2 = \omega^2\left(a\sqrt{1-\sin^2\omega t}\right)^2$$

$$v^2 = \omega^2 a^2(1-\sin^2\omega t)$$

$$v^2 = \omega^2 a^2 - \omega^2 a^2\sin^2\omega t$$

$$v^2 = \omega^2 a^2 - \omega^2 x^2$$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v = \pm\omega\sqrt{a^2-x^2}$$

(6) Let the distance of the rocket from the centre of the Earth be r

Let a be the acceleration away from the centre of the Earth

$$a = -g \times \frac{(6.4\times10^6)^2}{r^2}$$

$$\frac{d(\frac{1}{2}v^2)}{dr} = -9.8\times(6.4\times10^6)^2\, r^{-2}$$

$$\frac{1}{2}v^2 = 9.8\times(6.4\times10^6)^2\, r^{-1} + c$$

When $r = 6.6\times10^6$ $v = 5000$

$$12.5\times10^6 = \frac{9.8\times(6.4\times10^6)^2}{6.6\times10^6} + c$$

$$c = 12.5\times10^6 - \frac{9.8\times(6.4\times10^6)^2}{6.6\times10^6}$$

$$= -48319394$$

$$\therefore v = \frac{9.8\times(6.4\times10^6)^2}{r} - 48319394$$

~~When~~ The maximum height is reached when $v = 0$

$$\frac{9.8\times(6.4\times10^6)^2}{r} = 48319394$$

6 contd

$$r = \frac{9.8 \times (6.4 \times 10^6)^2}{48319394}$$

$= 830738.9$ m

$= 8307$ km

So the maximum height is $8307 - 6400$

$= 1907$ km

7. $a = -g - kv^2$

$$\frac{v\,dv}{dx} = -(g + kv^2)$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$\int dx = -\int \frac{v}{g + kv^2}\,dv$$

$$x = -\frac{1}{2k}\ln(g + kv^2) + c$$

$at\ x = 0\quad v = u$

$$0 = -\frac{1}{2k}\ln(g + ku^2) + c$$

$$c = \frac{1}{2k}\ln(g + ku^2)$$

$$\therefore x = \frac{1}{2k}\ln(g + ku^2) - \frac{1}{2k}\ln(g + kv^2)$$

$$x = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g + kv^2}\right)$$

max *height occurs at* $v = 0$

$\therefore$ *maximumheight of cork is* $\frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right)$

Year 12 Maths C Term 3 MPS Solutions

1.

$$xy\frac{dy}{dx} - (1+x^2)(1-y^2) = 0$$

$$xy\frac{dy}{dx} = (1+x^2)(1-y^2)$$

$$\frac{y\,dy}{(1-y^2)} = \frac{(1+x^2)dx}{x}$$

$$\int \frac{y\,dy}{(1-y^2)} = \int \frac{(1+x^2)dx}{x}$$

$$-\frac{1}{2}\ln(1-y^2) = \ln x + \frac{x^2}{2} + c \qquad \textit{note that} \frac{1+x^2}{x} = \frac{1}{x} + x$$

$$\textit{when } x = 1\ y = 0$$

$$\therefore\ -\frac{1}{2}\ln 1 = \ln 1 + \frac{1}{2} + c$$

$$c = -\frac{1}{2}$$

$$\therefore -\frac{1}{2}\ln(1-y^2) = \ln x + \frac{x^2}{2} - \frac{1}{2}$$

$$\textit{when } x = 2$$

$$-\frac{1}{2}\ln(1-y^2) = \ln 2 + 1.5$$

$$\ln(1-y^2) = -4.386$$

$$1 - y^2 = e^{-4.386}$$

$$y^2 = 1 - 0.01245$$

$$y = \pm 0.9938$$

2.

$let\ the\ thickness\ of\ ice\ at\ tim\ e\ t\ (hours) be\ h\ (cm)$

$$\frac{dh}{dt} = \frac{k}{h}$$

$$\int h\,dh = \int k\,dt$$

$$\frac{h^2}{2} = kt + c$$

$$when\ t = 0, h = 0 \therefore c = 0$$

$$\therefore \frac{h^2}{2} = kt$$

$$when\ t = 4\ h = 1$$

$$\frac{1}{2} = 4k$$

$$k = \frac{1}{8}$$

$$\therefore \frac{h^2}{2} = \frac{t}{8} \rightarrow \quad t = 4h^2$$

$$when\ h = 2.5\ t = 25$$

Therefore it takes 25 hours

Assumptions are that it begins at 0cm thickness, and temperature remains constant.

(3) At height x metres above the fixed magnet, the upward force on the movable magnet is $-0.2g + \frac{1}{2x^3}$

The acceleration is $-9.8 + \frac{5}{2x^3}$

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -9.8 + 2.5x^{-3}$$

$$\tfrac{1}{2}v^2 = -9.8x - 1.25x^{-2} + c$$

When $x = 2$ $v = 0$

$$\therefore \quad 0 = -19.6 - \frac{1.25}{4} + c$$

$$c = 19.6 + \frac{1.25}{4}$$

$$= 19.9125$$

$$\therefore \tfrac{1}{2}v^2 = -9.8x - \frac{1.25}{x^2} + 19.9125$$

It gets closest when $v = 0$

ie $\quad 0 = -9.8x - \frac{1.25}{x^2} + 19.9125$

Using the graphics calculator, the solution is

$$x = 0.269 \text{ m}$$

So the closest it gets is 0.269 m.

(4) a) This motion approximates simple harmonic motion and this is the model selected.

$a = 2 \quad , \quad p = 4$

$$\omega = \frac{2\pi}{p} = \frac{\pi}{2}$$

$$\ddot{x} = -\omega^2 x$$

The greatest acceleration will be at the bottom where $x = -2$

$$\text{Then } \ddot{x} = \frac{-\pi^2}{4} \times -2$$

$$= \frac{\pi^2}{2}$$

$$= 4.93 \ m/s^2$$

b) As the mass is 250g, the force will be

$$4.93 \times \frac{1}{4}$$

$$= 1.23 \ N$$

But gravity is also exerting 2.45N on the sausage and this needs to be balanced by tension in the spring. So the total tension will be

$$1.23 + 2.45$$

$$= 3.68 \ N$$

4 contd

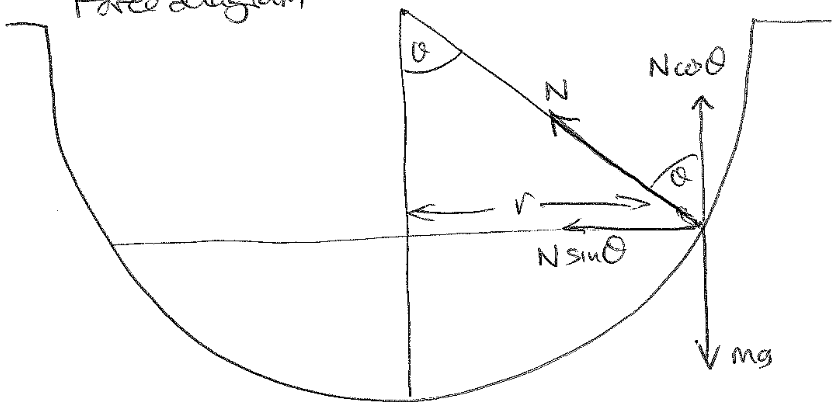
c) To generalise, let the sausage be of mass m kg

Then the tension is $4.93m + mg$

$$= (4.93 + 9.8)m$$

$$= 14.73m \text{ N}$$

(5) Force diagram



The cyclist's speed is 54 km/h = 15 m/s

$$N\cos\theta = mg \quad \ldots \text{(1)}$$

$$N\sin\theta = \frac{mv^2}{r}$$

$$N\sin\theta = \frac{mv^2}{20\sin\theta} \quad \ldots \text{(2)}$$

Dividing (2) by (1)

$$\tan\theta = \frac{mv^2}{20mg\sin\theta}$$

$$\tan\theta\sin\theta = \frac{v^2}{20g}$$

$$= \frac{225}{20\times 9.8}$$

$$= 1.148$$

Solving graphically for θ gives

$$\theta = 54.6^\circ$$

5 contd

$r = 20 \sin 54.6$

$= 16.30\,m$

Circumference of path $= 16.30 \times 2\pi$

$= 102.4\,m$

At 15m/s a lap takes $\frac{102.4}{15}$

$= 6.83\,s$