Calculus Basics

Functions

A relation is information which allows us to find the value of one quantity (variable) if we know the value of another quantity. A relation can be expressed as a set of ordered pairs, as a table, as a graph or, if it has a pattern, as a formula (equation).

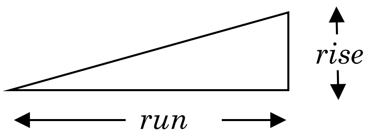
A function is a relation in which there is exactly one value for the dependent variable corresponding to every value for the independent variable. In Maths B (and Maths C) we will look at the calculus of functions only.

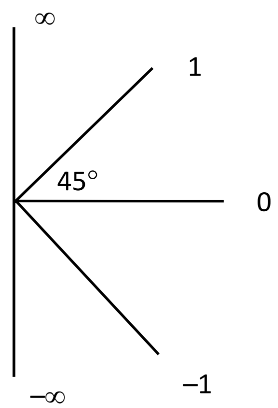
A function can be viewed as something you do to the independent variable. For instance the function $y = x^2$ is the function of squaring; $y = (x+2)^3 - 5$ is the function 'add 2, raise to the power of 3 then subtract 5'.

If x is the independent variable in a function and y is the dependent variable, then we say that y is a function of x. (It doesn't always follow that x is a function of y.) We can write y, a function of x, as $y(x)$. If $y = x^2$, we can write $y(x) = x^2$. $y(x)$ is then an operation performed on x, the operation of squaring. So $y(3)$ is the same operation performed on 3: $y(3) = 9$.

Gradient

The gradient of a line is the number of units the line goes up for each unit it moves to the right. It can be calculated as $\frac{rise}{run}$.

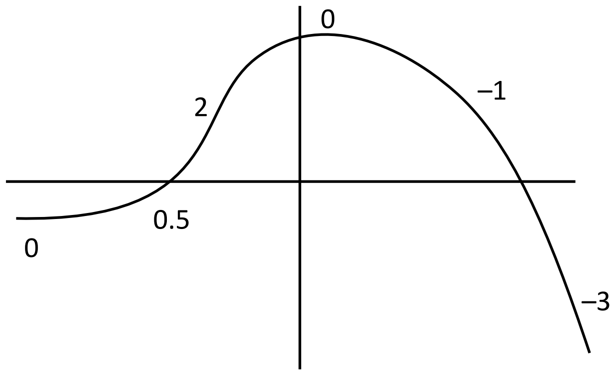




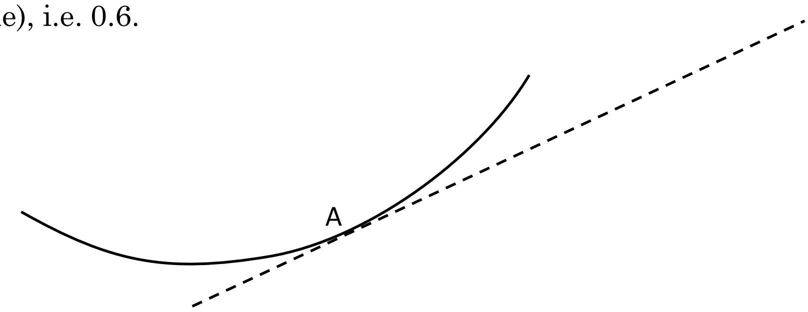
You should be able to estimate the gradient of a line by looking at it. 5 benchmark gradients are useful for this.

Gradient of a curve

A straight line has the same gradient at all points. On a curve, however, the gradient varies. For example, on the curve below, the gradient starts at zero, then increases to about 2, then decreases again back to zero and finally becomes negative.

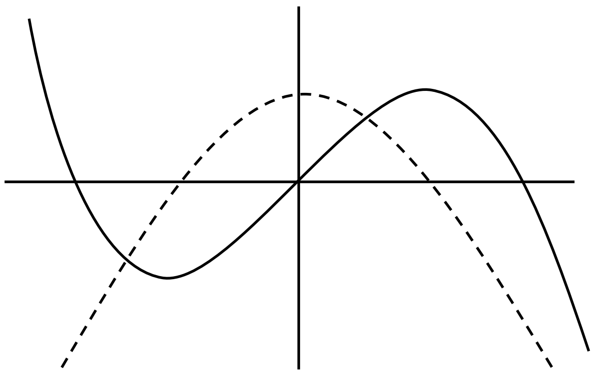


The gradient at a point on the curve is equal to the gradient of the tangent at that point. At point A on the curve (solid line) below, the gradient is equal to the gradient of the tangent (dashed line), i.e. 0.6.



Gradient Functions

For a given function, the gradient function is the relation between the gradient and the independent variable. For example, the function $y = x - \frac{1}{4}x^3$ is shown below as the solid line. [The x range is –3 to 3 and the y-range is –2 to 2.] To find the gradient function, pick a selection of points along the curve, estimate the gradient at each point and plot it on the same graph. Then join the points with a curve. The gradient function is shown as the dotted line.



The gradient function can be called the gradient function or the derived function or the derivative. For the function $y(x)$, the gradient function can also be called $y'(x)$ or $\frac{dy}{dx}$. The original function can be called the primitive function.

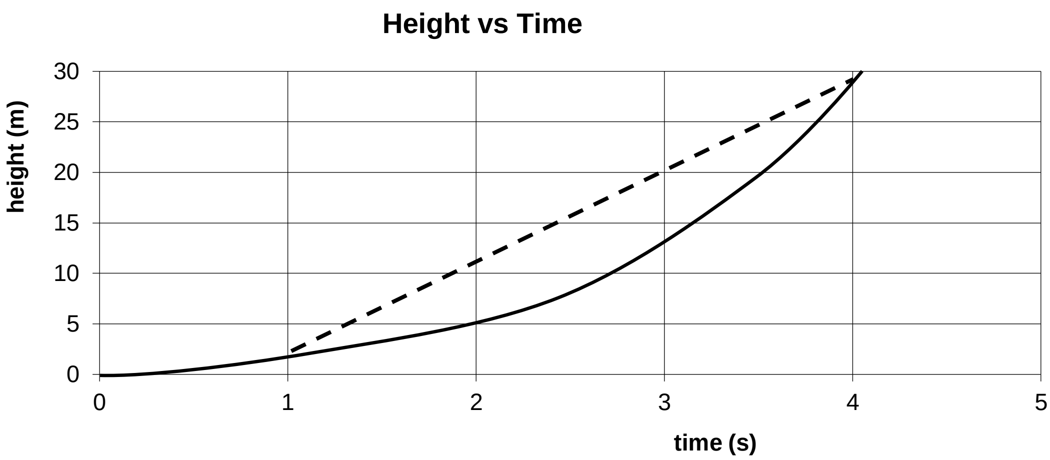
Finding the gradient function for a given primitive is called 'differentiating' or 'differentiating the primitive function'. You may hear people call it 'deriving the primitive function', but this is improper as it is the derived function that we derive, not the primitive function.

Gradient Functions with a Graphics Calculator

You should be able to graph a function on your graphics calculator and graph its gradient function on the same axes. You should also be able to find the y-value and the y'-value (gradient) for any x value.

Rate of Change

The gradient of a function indicates the rate at which the dependent variable is changing with respect to the independent variable. Suppose the solid curve on the graph below is the relation between height and time for a rocket. The independent variable is time in seconds and the dependent variable is height above the ground in metres. The gradient of the curve at any point is the rate of change of height with respect to time, i.e. the change in height per unit change in time. This will be measured in metres per second. This rate of change is one we use a lot, so we give it a name – velocity.



Average Rate of Change

In the relation above, let us call height h and time t. The average velocity over a given time interval (i.e. the average of rate of change of h with respect to t over that time interval) is given by the total change in h over that interval divided by the total change in t over that interval.

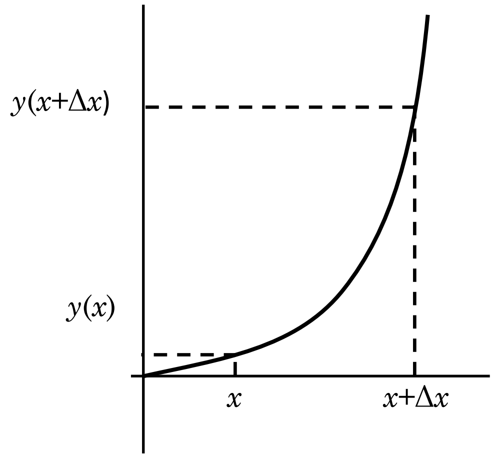
At t=1, h=2. At t=4, h=29. So the change in h between t=1 and t=4 is 29–2=27 and the change in t is 4–1=3.
Therefore the average velocity is 27÷3 = 9 m/s.

It is usual practice to call the change in height Δh (pronounced 'delta h' – Δ is a Greek d and stands for 'difference') and to call the change in time Δt.
We can then say that velocity $= \frac{\Delta h}{\Delta t} = \frac{27}{3} = 9$.

A straight line joining two points on a curve is called a secant. The average rate of change over an interval on the curve is the same as the average rate of change of the secant over that interval. This means it is the same as the gradient of the secant. The secant representing the average rate of change between t=1 and t=4 above is shown as the dashed line.

In a general relation between x and y, the average rate of change of y with respect to x over a given interval Δx is $\frac{\Delta y}{\Delta x}$.

Formula for Average Rate of Change



Consider a function $y(x)$. Suppose we want the average rate of change of y with respect to x over the interval from some value x to $x+\Delta x$. See graph at right.

This is represented by $\frac{\Delta y}{\Delta x}$ which is $\frac{y(x+\Delta x)-y(x)}{\Delta x}$.

For example, suppose we wanted the average rate of change of the function $y=x^2$ over the interval from x to $x+\Delta x$.

$\frac{\Delta y}{\Delta x} = \frac{y(x+\Delta x)-y(x)}{\Delta x} = \frac{(x+\Delta x)^2-x^2}{\Delta x}$. This can be simplified:

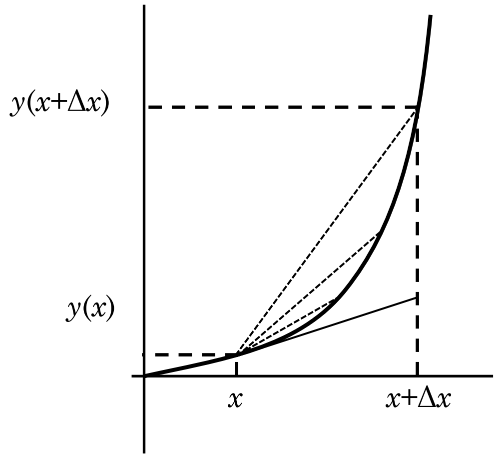
$$= \frac{x^2+2x\Delta x+\Delta x^2-x^2}{\Delta x}$$

$$= \frac{2x\Delta x+\Delta x^2}{\Delta x}$$

$$= 2x + \Delta x$$

[Note that Δx^2 means the square of Δx: Δx is a single variable, not a product of Δ and x.]

Instantaneous Rate of Change



The above allows us to find the average rate of change over an interval. But what if we want the rate of change at a particular instant (or at a particular value of x)?

Looking at the diagram again, the average rate of change between x and $x+\Delta x$ is given by the gradient of the secant (highest fine dashed line). The instantaneous rate of change is given by the gradient of the tangent (fine solid line).

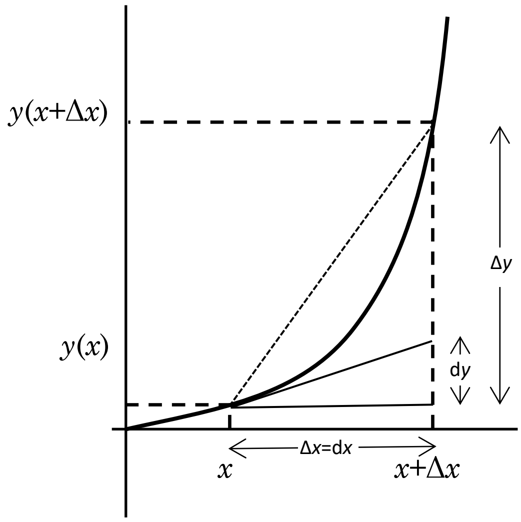
If we made Δx smaller and smaller, the gradient of the secant would get closer and closer to the gradient of the tangent (see the lower dashed lines). We cannot make Δx zero because then there would be no secant and $\frac{\Delta y}{\Delta x}$ would be $\frac{0}{0}$, which is not defined. But we can make it as small as we like.

We can calculate the gradient of the secant. As we saw above, for the function $y = x^2$, the gradient of the secant is $2x + \Delta x$.

As we make Δx smaller and smaller, the gradient of the secant gets closer and closer to $2x$. Thus we can say that the gradient of the tangent is $2x$ and that the instantaneous rate of change of $y=x^2$ at any value of x is $2x$. The gradient at $x=5$ is 10 and so on.

In the diagram to the right we define dy as being the rise of the tangent and dx as being the run of the tangent. $dx = \Delta x$, but $dy \neq \Delta y$. There is a limit to how small Δy can get as we make Δx smaller and smaller. This limit is dy. So we say that dy is the limit of Δy as Δx approaches 0, or, more commonly, that $\frac{dy}{dx}$ is the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches 0.

We can write this as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.



Differentiation by Rule

The above allows us to differentiate functions from first principles algebraically. There is a quicker way to differentiate called 'differentiation by rule' and this is the method you will use almost exclusively in the long run.

If $y = ax^n$, $\frac{dy}{dx} = nax^{n-1}$

In other words, to differentiate ax^n, we multiply by the index, then subtract 1 from the index.

Example: If $y = 5x^3$, $\frac{dy}{dx} = 15x^2$.

This works for any index including fractional and negative indices. Before differentiating functions like $y = \frac{4}{\sqrt{x}}$, we express it in the form ax^n. This function is $y = 4x^{-1/2}$. Then $\frac{dy}{dx} = -2x^{-3/2}$. We then change back to the original form of the function and write $\frac{dy}{dx} = -\frac{2}{\sqrt{x^3}}$.

If $y = \sin x$, $\frac{dy}{dx} = \cos x$

If $y = \cos x$, $\frac{dy}{dx} = -\sin x$

If $y = a^x$, $\frac{dy}{dx} = a^x \log_e a$ $\quad$ $e = 2.718281828459045\ldots$, an irrational number.

If $y(x) = f(x) + g(x)$, $y'(x) = f'(x) + g'(x)$

If $y = kf(x)$, $y' = kf'(x)$

If $y(x) = u(x) \times v(x)$, $y' = vu' + uv'$ (product rule)

If $y = \frac{u}{v}$, $y' = \frac{vu' - uv'}{v^2}$ (quotient rule)

If $y = f(g(x))$, $y' = f'(g(x)) \times g'(x)$ (chain rule)

Some of these rules take a lot of practice to get good at.

Anti-differentiation

If we know a derivative, it is possible to find the primitive. We can use a process of guess and check: think what kind of function might produce our derivative when differentiated. We write down a best guess. Then we differentiate it to check that it is the right function. If it isn't, we adjust it until it is right.

As an example, $\frac{dy}{dx} = 6x^2$ is a differential equation (an equation containing a derivative). To solve it, we might realise that the primitive will have x to the power of 3, so we might try $y = x^3$. This differentiates to $y' = 3x^2$. We need $y'=6x^2$, so we double our guess to $y = 2x^3$. This differentiates to $y'=6x^2$, so $y = 2x^3$ is correct.

We can lay this out like this:

$y' = 6x^2$
Try $y = x^3 \rightarrow y' = 3x^2$
Try $y = 2x^3 \rightarrow y' = 6x^2$ ✔
So $y = 2x^3 + c$

We have to add a constant c because when we differentiate a constant, we get zero, so any constant can be added to the primitive and it will still give the same derivative.

To find the value of c, we need to know the value of y for a given value of x. This information is called a boundary condition. This will allow us to write and solve an equation in c. For example, suppose we know that when x is 3, y is 12. Then we can continue:
$12 = 2 \times 3^3 + c$
$12 = 54 + c$
$c = -44$
So the particular solution of the differential equation is $y = 2x^3 - 44$.
$y = 2x^3 + c$ is called the general solution.

We often call the primitive of a function the integral of the function. The primitive of $6x^2$ can be called the integral of $6x^2$.
We write this as $\int 6x^2 dx$. So $\int 6x^2 dx = 2x^3 + c$.

Subsuper

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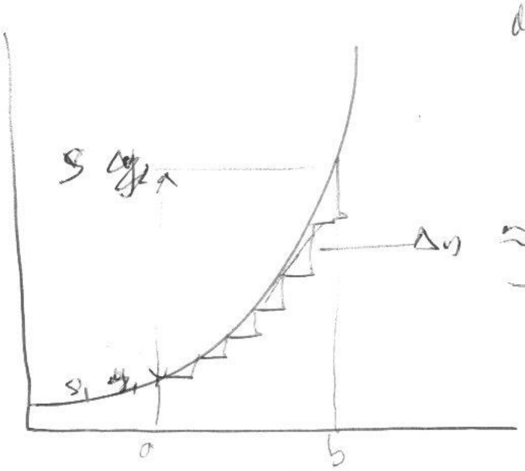
$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$



$\Delta y \approx v\,dx$

$\rightarrow v\,dx$ as $dx \rightarrow 0$

$\int \sum v\,dx$

As $dx \rightarrow 0$

$\Delta y \rightarrow dy = v\,dx$

$S_2 - S_1 = \int v\,dt$

In ~~limit~~ $S = \lim_{dt \to 0} \sum v\,\Delta t + S_1$

As $dx \rightarrow 0$ $\Delta y \rightarrow dy = v\,dx$ $= \int v\,dt + S_1$

$y_2 - y_1 = \lim_{dx \to 0} \sum v\,dx$

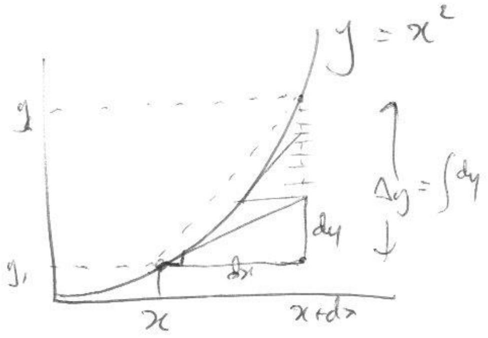
$= \int v\,dx$

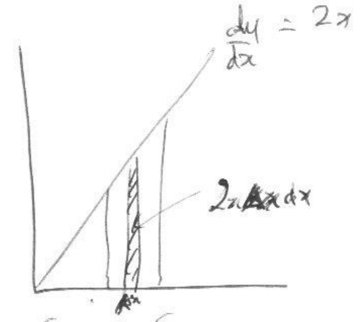
$S = \int v\,dt + S_1$

$S = \int v\,dt + S_1$

v is derivative, s is anti-derivative of v

So $\int v\,dt$ is antiderivative of v





$$\int \frac{dy}{\;} = \int 2x\,dx$$

$\sum dy$

$= \sum 2x\,dx$

Δy

$$y = x^2$$

$y_2 - y_1\ \Delta y = y(x+\Delta x) - y(x) \quad \sum \Delta y$

$= \lim_{\Delta x \to 0} \sum \Delta y$

$= \lim_{\Delta x \to 0} \sum dy$ [As $\Delta x \to 0$ $\Delta y \to dy$]

$= \lim_{\Delta x \to 0} \sum 2x\,dx$ [as $\frac{dy}{dx} = 2x$]

$= \int 2x\,dx$ [notation change only]

$y_2 = \int 2x\,dx = x^2 + y_1$

$= x^2 + C$