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Question 1

Show which of the four group properties are satisfied by the operation of subtraction on the set of all even integers {. . . –4, –2, 0, 2, 4, 6, . . . }.

Question 2

Draw a Cayley table for the set {1, 3} with the operation $\otimes_4$ (multiplication modulo 4). Given that the operation is associative, show whether this system forms a group.

Question 3

The operation ♥ is defined on the rational numbers by $a \heartsuit b = \dfrac{a+b}{ab}$. Find the identity or show that there isn't one.

Question 4

An economy consists of two industries, coal and steel. Making a tonne of coal requires 0.2 tonnes of steel and 0.4 tonnes of coal. Making a tonne of steel requires 0.5 tonnes of coal and 0.3 tonnes of steel. The final demand for coal and steel are 20 000 tonnes and 30 000 tonnes respectively. Use matrix methods to find the total demand for each.

Question 5

Evaluate $\begin{vmatrix} 2 & 0 & a & 0 \\ -1 & 1 & 4 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 5 & -1 & 1 \end{vmatrix}$, showing working.

Question 6

Find $\begin{bmatrix} a & 1 & 1 \\ 2 & -2 & 3 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$, showing working.

Question 7

In an arithmetic sequence $t_4 = -21$ and $t_9 = 4$. Find

(a) t_{55}
(b) the sum of the first 20 terms
(c) the number of terms required for their sum to exceed 5000

Question 8

Express the recurring decimal 0.23737373737…. as a common fraction in simplest form.

Question 9

$5+b$, 5 and $5 \div b$ are the first three terms (in that order) of a geometric sequence. Find the value of b.

Question 1

The system $[S, \times]$ forms a group. ($\times$ is normal multiplication.) The Cayley table for the group is shown unfinished below. Copy and complete it. Justify your choice of the fourth element in the set (the one marked with the arrow). No other working is needed.

×	1			
		–1		
			1	

←

Question 2

$1^2 = 1$ $1^2 + 2^2 = 5$ $1^2 + 2^2 + 3^2 = 14$ and so on.

The relation between s, the sum of the first n squares and n is a cubic polynomial of the form $s = an^3 + bn^2 + cn + d$. Write simultaneous equations and solve them using matrices to find the values of a, b, c and d. You should show the matrix equations, though the operations can be performed on a calculator.

Question 3

A drunk left the pub and walked a kilometre north, then half a kilometre south, then a quarter of a kilometre north, then an eighth of a kilometre south and so on.

Jose came up with a mathematical model to work out how long the drunk walked for. The model involved the sum to infinity of a geometric sequence.

a. Use Jose's model to work out how long the drunk walked for. There is one crucial parameter you will need to assume a value for. What is this parameter? Make a reasonable assumption as to its value, bearing in mind that the drunk was drunk.

Jose's mother, Maria, said that Jose's model was not a good one because it would take the drunk a second to turn around each time he changed direction.

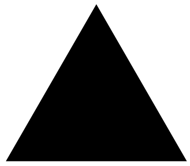
b. Adjust Jose's model to take into account Maria's concern and redo the calculations.
c. Does the new model give a reasonable result? Decide how it could be refined to make it more realistic? Justify your decision.

Question 4

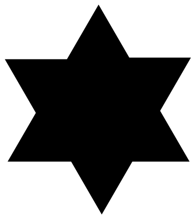
Find the sum of the first 326 terms of the sequence defined by
$t_n = t_{n-1} - t_{n-2}$ $t_2 = 16$ $t_4 = -5$

Question 5

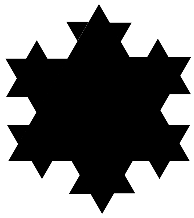
A Koch Snowflake is constructed like this:



- Draw an equilateral triangle with area 1 m^2



- Replace the centre third of each side with two lines both the length of the original third, like this:



- Replace the centre third of each side of the resulting figure with two lines both the length of the original third, like this:

- Repeat *ad infinitum* (for ever).

Find the perimeter and area of the resulting shape.

Comment on whether these results seem reasonable and, if not, why a seemingly unreasonable result may have arisen.

Year 11 Maths C Term 2 Knowledge and Procedures Exam Solutions

Q1

a) Closed because subtracting an even integer from an even integer will always produce an even integer.
b) Not associative because (8 – 5) – 1 = 2, but 8 – (5 – 1) = 4.
c) 0 would be the right identity because a – 0 = a, but, in general, 0 – a ≠ a. So there is no identity.
d) Because there is no identity, there can be no inverses.

Q2

$\otimes_4$	**1**	**3**
1	1	3
3	3	1

Every row and every column contain every element of the set. So it is closed with identity and inverse. Associativity is given. Therefore the system is a group
OR
Closure – the group is closed as all elements in the table are elements of the set {1,3}.
Associativity – Given as associative.
Identity – is 1, as the first column and row match the header column and row.
Inverse – in each row there exists the identity element and it also appears in the same position in each row, hence each element has an inverse.

Q3

Let the identity be u.
Then $a \heartsuit u = a$

$$\frac{a+u}{au} = a$$

$$a+u = a^2u$$

$$a = (a^2-1)u$$

$$u = \frac{a}{a^2-1}$$

As u is a variable, there is no identity.

Q4

Let c be the total demand for coal; let s be the total demand for steel.

$c = 0.4c + 0.5s + 20\,000$

$s = 0.2c + 0.3s + 30\,000$

$$\begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} 0.4 & 0.5 \\ 0.2 & 0.3 \end{pmatrix}\begin{pmatrix} c \\ s \end{pmatrix} + \begin{pmatrix} 20\,000 \\ 30\,000 \end{pmatrix}$$

$$\begin{pmatrix} c \\ s \end{pmatrix} - \begin{pmatrix} 0.4 & 0.5 \\ 0.2 & 0.3 \end{pmatrix}\begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} 20\,000 \\ 30\,000 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.5 \\ 0.2 & 0.3 \end{pmatrix}\right]\begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} 20\,000 \\ 30\,000 \end{pmatrix}$$

$$\begin{pmatrix} 0.6 & -0.5 \\ -0.2 & 0.7 \end{pmatrix}\begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} 20\,000 \\ 30\,000 \end{pmatrix}$$

$$\begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} 0.6 & -0.5 \\ -0.2 & 0.7 \end{pmatrix}^{-1}\begin{pmatrix} 20\,000 \\ 30\,000 \end{pmatrix}$$

$$\begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} 90\,625 \\ 68\,750 \end{pmatrix}$$

Therefore the total demand for coal is 90 625 t and the total demand for steel is 68 750 t.

Q5

$$\begin{vmatrix} 2 & 0 & a & 0 \\ -1 & 1 & 4 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 5 & -1 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 2 & 0 & a \\ -1 & 1 & 4 \\ -2 & 3 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 & a \\ -1 & 1 & 4 \\ -2 & 3 & 0 \end{vmatrix} \; expanded\ about\ first\ row$$

$$= 2\begin{vmatrix} 1 & 4 \\ 3 & 0 \end{vmatrix} - 0\begin{vmatrix} -1 & 4 \\ -2 & 0 \end{vmatrix} + a\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix}$$

$$= 2(0-12) - 0 + a(-3 - -2)$$

$$= -24 - a$$

Q6

$$A = \begin{bmatrix} a & 1 & 1 \\ 2 & -2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \begin{vmatrix} -2 & 3 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} a & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} a & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} & -\begin{vmatrix} a & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} a & 1 \\ 2 & -2 \end{vmatrix} \end{bmatrix}$$

$$C = \begin{bmatrix} -3 & 0 & 2 \\ 1 & 0 & -a \\ 5 & -3a+2 & -2a-2 \end{bmatrix}$$

$$adj\ A = \begin{bmatrix} -3 & 1 & 5 \\ 0 & 0 & -3a+2 \\ 2 & -a & -2a-2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \times adj\ A$$

$$= \frac{1}{2-3a}\begin{bmatrix} -3 & 1 & 5 \\ 0 & 0 & -3a+2 \\ 2 & -a & -2a-2 \end{bmatrix}$$

Q7

a) $t_4 = -21$

$-21 = a + 3d$

$t_9 = 4$

$4 = a + 8d$

Solve Simultaneously

$a = -36$ and $d = 5$

$\therefore t_{55} = -36 + (55-1) \times 5$

$= 234$

b) $S_{20} = \frac{20}{2}[2 \times -36 + (20-1) \times 5]$

$= 230$

c) $5000 = \frac{n}{2}[2 \times -36 + (n-1) \times 5]$

$5000 = -36n + \frac{5}{2}(n^2 - n)$

$0 = 2.5n^2 - 38.5n - 5000$

Solve using Quadratic formula or Graphics Calculator

$n = 53.08\ or - 37.7$

Reject negative solution

$\therefore$ 54 terms are required for the sum to exceed 5000

Q8

0.23737373737 … .

$= 0.2 + 0.037 + 0.00037 + 0.0000037 + \cdots$

$= \frac{2}{10} + \frac{37}{1000} + \frac{37}{100000} + \frac{37}{10000000}$

Sum of the terms $= \frac{2}{10} + S_\infty$

$= \frac{2}{10} + \frac{\frac{37}{1000}}{1 - \frac{1}{100}}$

$= \frac{2}{10} + \frac{37}{1000} \times \frac{100}{99}$

$= \frac{2}{10} + \frac{37}{990}$

$= \frac{198}{990} + \frac{37}{990}$

$= \frac{235}{990}$

$= \frac{47}{198}$

Q9

$r = \frac{5}{5+b} = \frac{5 \div b}{5}$

Cross Multiply

$25 = \frac{5}{b}(5+b)$

$25 = \frac{25}{b} + 5$

$20 = \frac{25}{b}$

$b = \frac{25}{20}$

$b = \frac{5}{4}$

Year 11 Maths C Term 2 Modelling and Problem Solving Solutions

Q1

×	1	i	–1	–i
1	1	i	–1	–i
i	i	–1	–i	1
–1	–1	–i	1	i
–i	–i	1	i	–1

←

The fourth element has to be –i because –i occurs in the body and the group must be closed.

Q2

$s = an^3 + bn^2 + cn + d$

When $n = 1$, $s = 1$, so $a + b + c + d = 1$

When $n = 2$, $s = 5$, so $8a + 4b + 2c + d = 5$

When $n = 3$, $s = 14$, so $27a + 9b + 3c + d = 14$

When n = 4, s = 30, so 64a + 16b + 4c + d = 30

As a matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 \\ 1/2 \\ 1/6 \\ 0 \end{pmatrix}$$

So $a = 1/3$, $b = ½$, $c = 1/6$, $d = 0$

Q3

a. The drunk walked 1 km + ½ km + ¼ km etc. This is a geometric sequence with a = 1, r = ½.

The sum to infinity is 1/(1–½) = 2. So he walked 2 km.

To find how long he walked for, we need to know his average speed. We will assume he walked at 2 km/h. Thus he would have walked for 1 hour.

b. If he took 1 s to turn around, then the time taken in seconds would be 1800 + 1 + 900 + 1 + 450 + 1 …

The sum to infinity is now infinity.

c. This is not a reasonable result as no one can walk for ever, let alone a drunk. It could be made more realistic by not doing an infinite number of turns, but instead assuming that when he is travelling less than 1 cm, he stops. The number of walks before he travels less than 1 cm would be given by 100 000 ÷ 2^n = 1, i.e. 2^n = 100 000. n = $\log_2$ 100 000 = 16.6. So he would do 17 walks. The time taken for this is the sum of the geometric sequence to 17 terms plus 17 seconds.

S_{17} = 1800 (1 – 0.5^{17}) ÷ (1 – 0.5) + 17 = 3617 s
This modification is justifiable in that once the drunk is moving less than a centimetre, he is essentially still and can be considered to have finished walking.

Q4
$t_2 = 16 \; and \; t_4 = -5$
$t_4 = t_3 - t_2$
$-5 = t_3 - 16$
$t_3 = 11$
$t_3 = t_2 - t_1$
$11 = 16 - t_1$
$t_1 = 5$

the sequence is $5, 16, 11, -5, -16, -11, 5, 16, 11, -5, -16, -11$
The pattern repeats itself after 6 terms, the sumn of those 6 terms is 0
$\therefore 326 \div 6 = 324$ remainder 2
Therefore the sum of the first 326 terms is the same as the sum of the first 2 terms
Hence $S_{326} = 21$

Q5
Perimeter
Side length = a m
1st stage $P = 3a$
2nd Stage $P = 3a \times \frac{4}{3}$
$\therefore$ after infinite iterations $= 3a \times \left(\frac{4}{3}\right)^{\infty}$
$P = \infty$
Area
1st stage $A = 1m^2$
2nd stage $A = 1 + 3 \times \frac{1}{9} = 1 + \frac{3}{9}$
3rd stage $A = 1 + \frac{3}{9} + 12 \times \frac{1}{81} = 1 + \frac{3}{9} + \frac{12}{81}$
4th stage $A = 1 + \frac{3}{9} + \frac{12}{81} + 48 \times \frac{1}{729} = 1 + \frac{3}{9} + \frac{12}{81} + \frac{48}{729}$
Hence a GP exists after 1
$a = \frac{3}{9}, r = \frac{4}{9}$
$A = 1 + S_{\infty}$
$A = 1 + \frac{\frac{3}{9}}{1 - \frac{4}{9}}$
$A = 1\frac{3}{5}m^2$