How Functions Grow

This lesson plan is especially targeted at one of the CAM standards, namely the Grades 11 and 12 Benchmark (CIM and CAM) "Representations of Mathematical Relationships." Questions for students are printed in red.

Functions That Get Big

In this lesson, we want to study functions that "get big" - functions f(x) for which the function values f(x) grow without limit as x get larger and larger. What does this mean in terms of the graph? What are some examples of functions with this property? Get the students to list as many as possible. Here are some functions you might have on your list: f(x) = x , x2 , x3, xn (for which values of n?) ex , 2x, ax (for which values of a?) ln(x), log10(x) and many, many more. It is a good activity in class to make quite a long list. This topic is often first encountered in a calculus class, but there is no good reason to introduce any limit notation here - the concept can be made quite clear just thinking about the graph of f(x) having to cross all horizontal lines no matter how high up they are.

How to Compare Rates of Growth

It is quite common in everyday language to compare rates of growth. For example, according to a theory of Thomas Malthus, food production tends to grow at an arithmetic (linear) rate and population tends to grow at a geometric (exponential) rate. These rates are very different. If both were growing at different linear rates, the difference would not be so striking - in fact, we could describe the difference as proportional. The difference between a linear and an exponential rate is quite striking and we could say that the two rates are out of proportion. An observation we could make here is that we often compare two growing quantities by examining their ratio. If that ratio is constant, we say the two quantities are proportional (in fact, we say they are proportional even if the ratio is just roughly constant.) Here is a definition that we can use to compare rates of growth of functions: Take two functions from our list of functions that "get big", call them f(x) and g(x). If the ratio f(x)/g(x) is constant, we say the two functions are proportional. If the ratio f(x)/g(x) continues to grow (in other words if the ratio f(x)/g(x) is another function that "gets big") they we should say that the numerator function f(x) grows significantly faster than the denominator function g(x). Here are some examples:

1. Take f(x) = 2x and g(x) = 3x. Although g(x) gets large faster than f(x), we still say they are proportional since the ratio f(x)/g(x) = 3/2 is a constant. Can you generalize this to any pair of linear functions?

2. Take f(x) = x2 and g(x) = 10x. These are two functions that "get big" so lets compare these two functions by taking their ratio : f(x)/g(x) = x/10 . Since this is a function that "gets big" we say that the numerator function x2 "grows significantly faster" than the denominator function 10x. Next you might try f(x) = 1/10 x2 compared to g(x) = 1000 x. What generalizations can you make based on examples like these?

3. Repeat the previous example for pairs of functions like f(x) = x3 and g(x) = 5x2 and f(x) = x8 compare to g(x) = 6 x5. What generalizations can you make based on examples like these?

4. Next you might try the case when f(x) and g(x) are both polynomials. A standard "trick" for comparing these functions if to divide the numerator and denominator of f(x)/g(x) by the highest power of x that appears. For example, in the ratio (x3 + 3x)/(3x2 + 5x - 10) divide the top and bottom by x3 to get (1 + 3/x2) / (3/x + 5/x2 - 10/x3). Now observe that as x gets large, the numerator is approximately equal to 1 and the denominator is very small. This makes the ration very large.

Comparing Rates of Growth Using a Graphing Calculator

There are a couple good ways to compare two functions that "get big" using a calculator still using our basic idea of comparing the two functions by examining their ratio. If f(x) and g(x) are are two functions we could either make a table of values for f(x)/g(x) or simply look at the graph of f(x)/g(x). What property of the graph would tell us that f(x) grows significantly faster than g(x)? (The graph would keep going up.) What property of the graph would tell us that g(x) is the one that grows significantly faster? (The graph would tend to zero as x gets large.) What property of the graph would tell us that f(x) and g(x) are roughly proportional? (The graph of the ratio would have a horizontal asymptote at some y-value different from 0.) Here are some examples and exercises you could try with students.

1. Examine the case when f(x) and g(x) are both polynomials using the graphing method. Come up with rules for when f(x) grows faster and when they are proportional. (The polynomial with the higher degree grows faster. They are roughly proportional when they have the same degree.)

2. Compare an exponential function, say f(x) = ex to g(x) any polynomial using the graphing method. Can you make the polynomial grow faster than f(x)?

3. Compare a logarithmic function to polynomials.

4. Compare different exponential functions to one another.

5. Compare different logarithmic functions to one another.

Off to the Races! Building a Heirarchy of Functions.

Lets group our functions (remember, we are only considering functions that "get big") according to their rates of growth. One way to do this is to put functions together into the same "family" if their rates of growth are roughly proportional. We could say that one "family" of functions grows faster than another "family" if that is true of the members of the "families". For example, all linear functions would belong to the "linar family" and all quadratic functions would belong to the "quadratic family" but we would have to say that the "quadratic family" functions grow faster that the "linear family" functions. A great exercise is to let the students come up with as many families as possible and to rank them according to their rates-of-growth. The families and the rankings of the functions we have come up with so far should look like this (I use the symbol "<" to mean "grows slower than" - a good idea is to let the students invent notation for this.)

Families lister from slowest to fastest: logarithic functions < linear functions (polynomials of degree 1) < quadratic functions (polynomials of degree 2) < polynomials of degree 3 < ... < polynomial of degree n < polynomials of degree (n+1) ...< exponential functions.

How to Make a Faster Function (or a Slower One)

When students meet this idea they are usually interested in finding functions that grow really fast. Here are some ideas for questions to look at in class or for homework:

1. Find a function that grows even faster than an exponential function

2. Find a function that grows even faster than the one you found in (1). (etc. etc. etc.)

3. Find a function that grows slower than any logarithmic function. (and slower than than etc. etc. etc.)

4. Find a function that grows slower than any linear function, but faster than any logarithmic function.

5. Find a function that grows faster than any polynomial but slower than any exponential function.

6. Where does the sequence of numbers n! fit into this scheme? (They don't come from a function defined for all real numbers until you have invented the Gamma function, but we can still look at them just for positive integers.)