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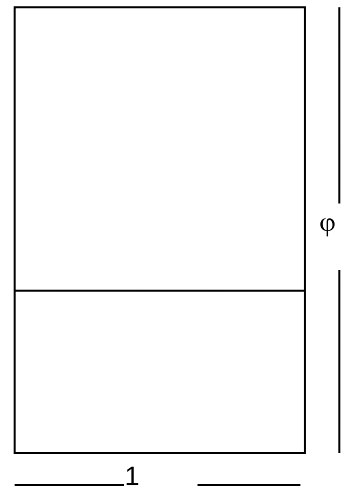
Mozart and the Golden Ratio

Amadeus Mozart composed some of the most beautiful music known to humankind. His music is revered for melody, but also for its "most exalted proportions" (Hanns Dennerlein). The proportion that is the best candidate for "most exalted" surely would be the golden proportion (or golden ratio), which is found in many works of art and architecture. A famous example is the Parthenon, from ancient Greece, which contained many instances of the golden proportion.



1. Consider the golden rectangle in the diagram alongside. A golden rectangle has side lengths whose ratio of lengths gives the golden proportion. If the short side of the rectangle has length 1, then the longer side equals the golden proportion, which is often denoted by the Greek letter φ.

   You can determine the value of φ as follows. An interesting and useful fact is that if a square is cut from a golden rectangle, the remaining shape is also a golden rectangle (and hence similar to the original rectangle).

a. Use the above information (and your knowledge of similar shapes and quadratic formulas) to find the exact value of φ.

b. Show that the value of *(p* to 2 decimal places is 1.62.

John F. Putz, a mathematician at Alma College, was told by his son that Mozart's piano sonatas are divided into two distinct sections." Putz recalled, "I knew that Mozart's music is highly regarded for its elegant proportions, among other things, so 1 thought it would be interesting to check whether the divisions Mozart used were very close to golden-section divisions." Here we will take the golden section to be the reciprocal of T.

2. Write the value of φ to 6 decimal places. Now find the reciprocal of φ to 6 decimal places. You should see something interesting. Explain what you have discovered.

In Mozart's time, a sonata was written in two parts, the Exposition, where the theme is introduced, and the Development and Recapitulation, where the theme is developed. Putz wondered if there was a relationship between the length of the Exposition and the length of the Development and Recapitulation. The length of a part is given by the number of measures in that part.

Mozart wrote 29 sonatas that had this two part structure. In the table below, the sonatas are listed by their Köchel number, part a is the length of the Exposition and part b is the length of the Development and Recapitulation.

For example, the first movement, which has Köchel number 279, has 38 measures in the Exposition, so $a = 38$, and 62 measures in the Development and Recapitulation, so $b = 62$. The length of the sonata is represented by $a + b$, or 38 + 62 = 100 measures. The ratio of $\frac{b}{a+b} = \frac{62}{100} = 0.62$. This is the golden section, rounded to the nearest hundredth.

3. Is this a coincidence, or did Mozart largely compose his sonatas to fit this pattern? Consider this table, which contains the information on the 29 sonatas that is needed to explore this question.

Kochel	***a***	***b***	***b + a***
279, I	38	62	100
279, II	28	46	74
279, Ill	56	102	158
280, I	56	88	144
280, II	24	36	60
280, III	77	113	190
281, I	40	69	109
281, II	46	60	106
282, I	15	18	33
282, III	39	63	102
283,I	53	67	120
283,II	14	23	37
283, III	102	171	273
284, I	51	76	127
309, I	58	97	155
311, I	39	73	112
310, I	49	84	133
330, I	58	92	150
330, III	68	103	171
332, I	93	136	229
332, III	90	155	245
333, I	63	102	165
333, II	31	50	81
457, I	74	93	167
533, I	102	137	239
533, II	46	76	122
545, I	28	45	73
547a, II	78	118	196
570, I	79	130	209

If Mozart was consistently using the golden section in his sonatas, what would you expect to find if you constructed a scatterplot of *b* versus a+ *b?*

4. Construct the scatterplot, and find the r^2 value and the equation of the least squares regression line. Comment on how this scatterplot matches your answer to the previous question.

5. On the same set of axes, construct the line $y = \varphi x$. How well does the data fit this line?

6. Construct a residual plot based on this model. Comment.

7. Construct a histogram of the values of $\frac{b}{a+b}$. Comment.

8. This activity is based on an article in the Mathematics Magazine (Vol. 68, No. 4, October 1995, p. 275, ff.) In the article, Putz states that if a movement is divided exactly in the golden section, then both $\frac{a}{b}$ and $\frac{b}{a+b}$ should be equal to φ. Show that this is true.

9. The above implies that an alternative approach to this problem is to plot a versus b. Do this, and find the value of r^2 and the equation of the least squares regression line. What do you find? Is this surprising?

10. There is a theorem that holds for values of a and b which states that $\frac{b}{a+b}$ is always nearer to φ than $\frac{a}{b}$. Choose 10 values for a and b (where $a < b$) and confirm this for yourself

11. Based on the above, which ratio should the researcher use: $\frac{b}{a+b}$ or $\frac{a}{b}$. Explain.

12. Some think that Mozart deliberately composed his sonatas so they related to the golden section, while others believe this pattern is just a coincidence. Others believe that Mozart incorporated the golden section in his sonatas subconsciously. What is your opinion? Justify your answer.