Relations, Equations and Functions – A Summary

Relations and Equations

1 A relation is information which allows one to find the value of one quantity if one knows the value of another quantity. Relations can be expressed as sets of ordered pairs, as tables and as graphs.

2 The quantities in a relation can take on various values. So we call them variable quantities or just *variables*. We can change a relation from one form to another. This requires knowing which is the independent variable and which is the dependent variable. The dependent variable is the one that is usually worked out from the independent variable. The dependent variable is:
- the second number in each ordered pair
- in the bottom row or right column of a table
- on the vertical axis of a graph

3 Where a relation has a pattern, it can also be expressed as a formula. The variable written by itself on the left side of the = sign is the dependent variable.

4 Some relations are discrete and some continuous. This depends on whether the *independent* variable is discrete or continuous. A continuous variable can take on any value within a range; a discrete variable can take on certain values, but there are values between them that it cannot take. The number of people in a family is a discrete variable because it cannot be say 3.27. People's heights is a continuous variable. It can be 175.5 cm or 175.6 cm or anything in between.

5 If we substitute a number for one of the variables in a formula we get a solvable equation. If we substitute for the independent variable, we get an explicit equation. For example if we substitute $d = 5$ into $f = d \times 3 + 2$, we get $f = 5 \times 3 + 2$, which can be solved with arithmetic.

6 We can convert between formulae, tables and graphs for a linear relation, but we must ensure that the independent variable in the original form remains the independent variable in the new form.

7 Different formulae can represent the same relation. For instance $h = 2(c + 4)$ and $h = 2c + 8$ represent the same formula: whatever value we pick for c, we will always get the same value for h whichever formula we use; both formulae will produce the same set of ordered pairs, the same table and the same graph. We can check by substitution whether two formulae are equivalent.

8 If we substitute for the dependent variable in a formula, we get an implicit equation. For example if we substitute $f = 14$ into $f = d \times 3 + 2$, we get $14 = d \times 3 + 2$. An implicit equation can be solved by guess and check, backtracking or balance.

9 Some formulae can have more than one independent variable. To get a solvable equation in this case, we substitute for all but one of the variables.

10 As well as being produced from a formula, solvable equations can be formed directly from a problem situation without explicit use of the relation.

11 Solving by balance may require the additional techniques:
- collecting terms
- expanding brackets
- operating with the unknown
- deciding what to use as the unknown

12 Formulae can be rearranged to make a different variable the dependent one (or subject).

Functions

13 A function is a relation in which for each value of the independent variable there is just one value for the dependent variable. To tell if a relation is a function:
- as ordered pairs or tables, no two ordered pairs have the same first number with different second numbers
- as a graph, no two ordered pairs on the graph lie in the same vertical line
- as a formula, the dependent is not raised to an even power and the formula does not contain a $\pm$

14 A function with a formula can be thought of as a sequence of operations to be performed on the value of the independent variable to obtain the value of the dependent variable. For instance the function $p = r \times 3 + 2$ can be thought of as 'multiply by 3, then add 2' or as '$\times 3 + 2$'.

15 We often use the shorthand f, g, h etc. to specify a function, so we might say the function f is $\times 3 + 2$ and the function g is $\times 5 - 4$. f and g specify the sequence of operations and we put a number in brackets after the letter to say what number we are applying the operations to. So $f(10) = 10 \times 3 + 2$; $g(a) = a \times 5 - 4$. It is normal practice, however, not just to write $f = \times 3 + 2$ because this is not grammatical; instead we generally write $f(x) = x \times 3 + 2$, where x can represent any number or variable. So, if $f(x) = x \times 3 + 2$, then $f(6) = 6 \times 3 + 2$ and $f(c^2) = c^2 \times 3 + 2$. Note that, if $f(x) = x^2 + 3x + 2$, then $f(10) = 10^2 + 3 \times 10 + 2$: if x occurs more than once in the formula, we substitute 10 for each occurrence of x.

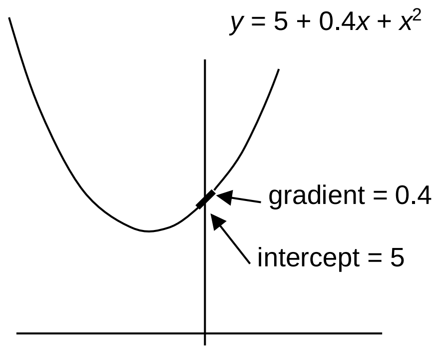
16 Commonly used functions can be divided into several types or families. Each type has a particular form for the formula (called the general form) and a particular shape for the graph.

17 Linear functions have the general form $y = a + bx$. Note that when we are talking about functions without specifying whether the variables are distance, time, cost, mass or whatever, we generally call the independent variable x and the dependent variable y. So in this case, the variables are x and y. a and b are called parameters. Parameters are numbers which are fixed in a particular relation, but which take on different values in different relations of the same family. So in one linear function,

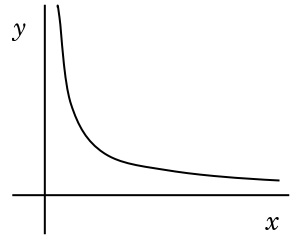
a might be 4 and b might be 5, making the formula $y = 4 + 5x$. In another linear function, a might be 2 and b might be –3, making the formula $y = 2 - 3x$.

18 The graph of a linear function is always a straight line. The parameter a specifies the starting value or y-intercept; the parameter b specifies the steps the dependent variable goes up in each time the independent variable goes up 1, in other words the gradient of the line. The relation between taxi fare and distance travelled is a linear function. a is the flag fall and b is the cost per kilometre .

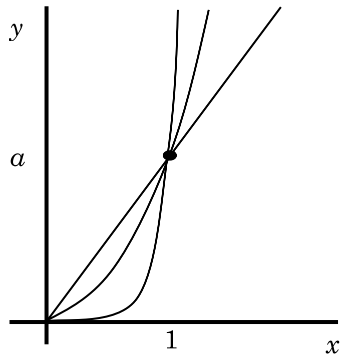
19 A quadratic function has the general form $y = a + bx + cx^2$. It crosses the y-axis at a with gradient b, but then curves up if c is positive, down if c is negative. This produces a parabola. The relation between height and time when one throws a stone is a quadratic function. a is the height from which it is thrown, b is the upward speed at which it is thrown and c is the amount of acceleration produced by gravity. (c will be negative in this case because the acceleration is downwards.)



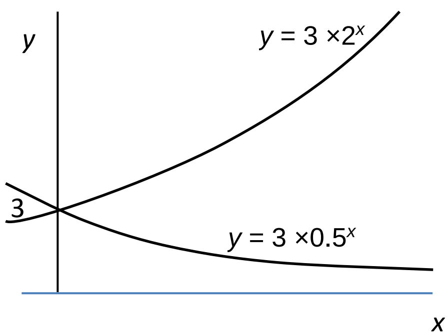
20 A reciprocal function has the general form = $\frac{k}{x}$, where k is the parameter. Its shape is like the picture to the right. The higher the value of k, the further the curve is from the origin. The relation between two variables which are in inverse proportion is a reciprocal function. An example is the relation between the time it takes to clean a stadium and the number of cleaners doing it.



21 A power function has the general form $y = ax^b$. The graphs look like those to the right for $b = 1, 2, 3$. The straight line is for $b = 1$, the one with the sharpest bend is for $b = 3$. All graphs pass through the point $(1, a)$.



22 An exponential function has the general form $y = ab^x$. It is called exponential because x is the exponent. The graphs look like those to the right, rising as you move to the right if $b > 1$ and falling if $b < 1$. They cross the y-axis at $y = a$. Exponential function model situations where the dependent variable is multiplied by the same number each time the independent variable increases by 1. The relation between money in the bank and time



under compound interest is an exponential function.

Relations and Equations – Questions to Check Understanding

1.1 The relation below is given as a set of ordered pairs.

(0, 125), (1, 108), (2, 102), (3, 94), (4, 94), (5, 89), (6, 90)

The first number is the age of a child in years, the second number is their heart rate in beats per minute. Use it to find:

(a) the heart rate when the child was 4

(b) the age at which the heart rate was 108

1.2 The relation below shows, as a table, average life expectancy for females of various ages in 17th Century England.

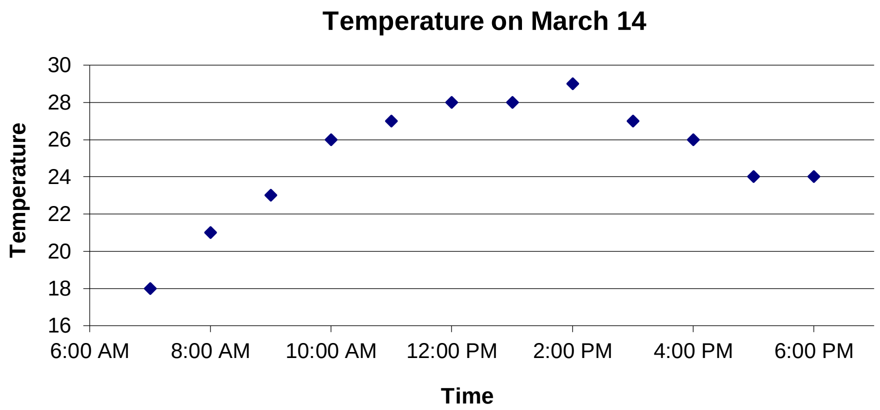
Age	0	20	40	60	80
Life expectancy	36	49	64	70	85

Use it to find:

(a) the life expectancy of a 20-year old

(b) the age you have to reach before your life expectancy is 64

1.3 The relation between temperature and time for March 14 is shown below.



(a) What was the temperature at 9 a.m.?

(b) When did the temperature first reach 27°?

2.1 For the relation between age and heart rate in Q1.1,

a. what is the independent variable and what is the dependent variable?
b. express the relation as a table
c. express the relation as a graph

2.2 For the relation between age and life expectancy in Q1.2,

a. what is the independent variable and what is the dependent variable?
b. express the relation as a set of ordered pairs
c. express the relation as a graph

2.3 For the relation between temperature and time in Q1.3,

a. what is the independent variable and what is the dependent variable?
b. express the relation as a set of ordered pairs
c. express the relation as a table

3.1 For each of the following relations, say whether it has a pattern and, if it does, find the next two ordered pairs.

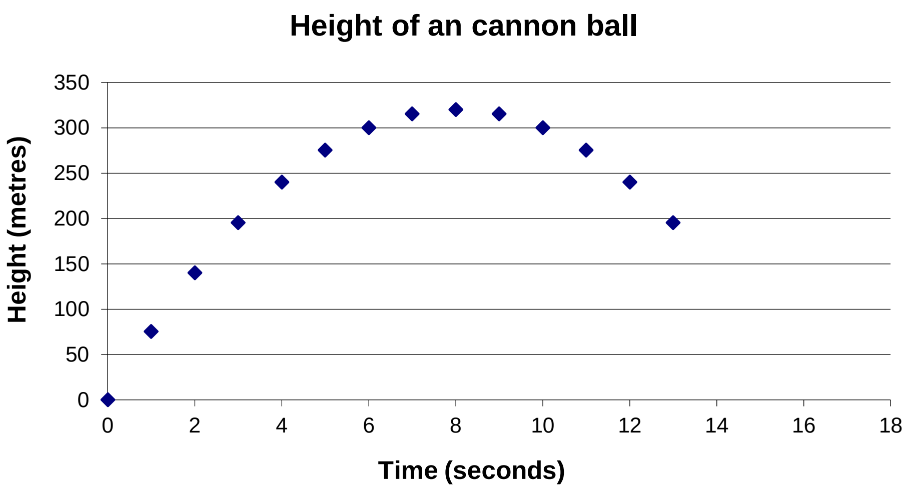
(a)

Age	0	10	20	30	40	50	60	70
Mass (kg)	3	13	22	31	37	41	48	54

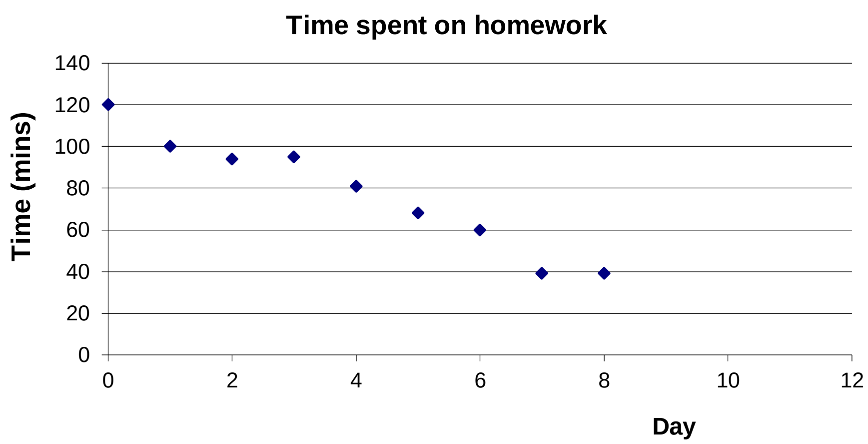
(b)

Number ordered	1	2	3	4	5	6	7
Price	5	6.5	8	9.5	11	12.5	14

(c)



(d)



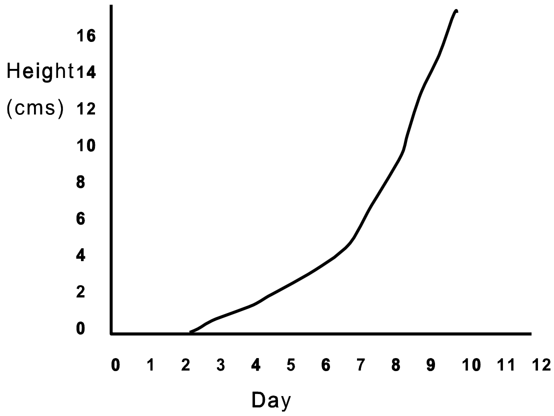
(e) How can a relation with a pattern be expressed other than as a set of statements, set of ordered pairs, table and graph?

4.1 For each of the following relations, state whether it is discrete or continuous.

(a) The results of rolling a die 50 times:

Result	1	2	3	4	5	6
Frequency	10	7	9	5	10	9

(b)



4.2 You should never join points on a graph by lines when plotting discrete data. Explain why.

5.1 Mawler is a hit man. He charges for jobs according to the time taken. He uses the following formula:

charge = 2500 + 400 × *time*,

where the *charge* in dollars and the *time* is hours.

How much would he charge for a job that took 3½ hours?

5.2 Use the formula *mass* = 60 + *age* × 2 to find the height if the mass if the age is 40.

6.1 Find formulae for these relations.

(a)

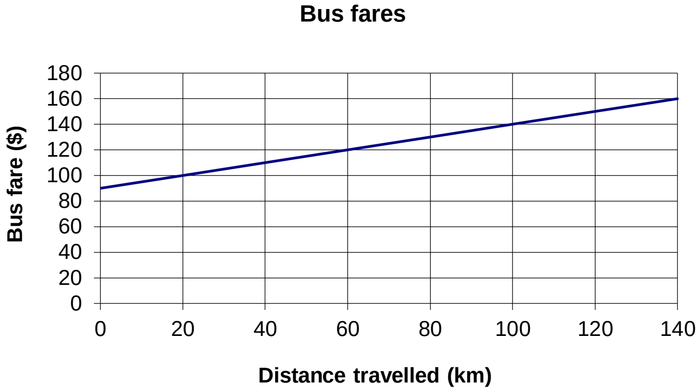
Distance (km)	1	2	3	4	5	6
Fare ($)	7.50	11.50	15.50	19.50	23.50	27.50

(b)

Length of visit (h)	½	1	1½	2	2½	3

Fee ($)	40	55	70	85	100	115

(c)



6.2 It has been discovered that the number of passengers carried on a particular route is related to the number of buses that run per day. The number of buses can vary from 6 to 12. The relation can be expressed by the formula

number of passengers = 70 + *number of buses* × 8

Express this relation as a table and as a graph.

7.1 Find out which of the following pairs of formulae are equivalent and which are not.

a) *price* = 12 + *number* × 6 *price* = (*number* + 2) × 6

b) *length* = *number of links* × 5 – 3 *length* = (*number of links* – 3) × 5

c) *capacity* = 3 + 8 × *length* *capacity* = (2 × *length* + 2) × 4 – 5

d) *pay* = 16 × *number of hours worked* + 18
pay = 8 + (*number of hours worked* + 1) × 8

8.1 (a) The formula for the simple interest paid on an investment is

interest = *principal* (in dollars) × *rate* (in %) × *time* (in years) ÷ 100.

Find the interest paid if: *principal* = $500, *rate* = 12%, *time* = 3 years

(b) *distance travelled* = *initial speed* × *time* + ½ × *acceleration* × $time^2$

Find the distance travelled if the acceleration is 4, the time is 10 and the initial speed is 20.

9.1 Solve the following by guess and check. Show working.

(a) If *number of matches* = *number of squares* × 3 + 1, find the number of squares if the number of matches is 55

(b) If *mass* = *length* ÷ 3 – 8, find the length if the mass is 8

(c) If *value* = $mass^2$ + *mass* × 4 + 12, find the mass if *value* = 400 (answer correct to 1 decimal place)

(d) If *dosage* × 10 + 15 = 66, find *dosage.*

9.2 Solve the following by backtracking. Show working.

(a) If *number of matches* = *number of caltros* × 3 + 7, find the number of caltros if the number of matches is 49

(b) (*speed* + 5) × 4 – 13 = 51

(c) 27 + 5 × (*height* – 11) = 101

9.3 Solve the following using balance. Show working.

(a) If *fee* = *hours worked* × 25 + 45, find the hours worked if *fee* = 195

(b) (*time* + 10) ÷ 2 + 28 = 143

(c) *s* × 4 – 18 = 16

(d) 4(2*a* + 5) = 108

(e) $\frac{3h+5}{6} = 43$

10.1 Solve the following by writing and solving an equation. Show working.

(a) Gonzo thought of a number, added 6, multiplied the result by 3, then subtracted 13. This gave him 71. What number did he first think of?

(b) If Harry had twice as much money and another $20, he would have $87. How much does he have?

(c) If Fiona gave me 30% of her beans, she would give me 63. How many does she have?

(d) Prudence is 12 years older than Randy, who is half as old as Dick. If Prudence is 36, how old is Dick?

11.1 Solve the following by writing and solving an equation. You will need to collect terms. Show working.

(a) Connie picked a number, multiplied it by 7, subtracted 8, then added the number she started with, then added 4. If he ended up with 60, what number did he start with?

(b) Bazza got 6 hours pay plus a $10 tip. He then spent 2 hours pay and $26, leaving him with $22. How much did he get paid per hour?

(c) Monolith had a packet of Minties with 5 missing. Clatterbones had two packets (the same as Monolith's packets) plus 114 loose ones. They had 166 Minties between them. How many Minties in a packet?

11.2 Solve the following. You will need to expand brackets. Show working.

(a) 2(*m* + 6) + 3(*m* + 3) = 41

(b) Cowboy thought of a number, added 5, then multiplied the result by 4. He then added the number he started with. This gave him 80. What number did he start with?

(c) Slimey had some slugs. On Friday she had 6 more than on Thursday. On Saturday she had 4 times as many as on Friday. On Sunday she had 12 less than on Saturday. On Monday all the ones she had on Thursday had escaped leaving her with 51. How many did she have on Thursday?

11.3 Solve the following. You will need to operate with the variable. Show working.

(a) $\mathbf{4n + 5 = 25 - n}$

(b) $\mathbf{38 - 5a = 22}$

(c) $\frac{18}{x} + 11 = 0$

(d) $(2b + 32) \div b = 10$

(e) Ginny thought of a number, added 3, then multiplied by 5, then subtracted 8. He ended up with 4 more than twice the number he started with. What number did he start with?

(f) Jonno had enough money to buy 3 oosalem birds. He then spent $40, then lost ¾ of his money in a bet, then earned another $155 , then bought one oosalem bird. This left him $72 less than he would need to buy 2 more oosalem birds. How much does an oosalem bird cost?

11.4 Solve the following. You will need to decide which quantity to use as the unknown. Show working.

(a) Lucy worked for 4 days and earned twice as much each day as on the previous day. She also got a $100 bonus at the end, making her total pay $278.50. How much did she earn on the third day?

(b) Lenin had an average number of customers on Monday. He had 5 more than average on Tuesday. On Wednesday he had twice as many as Tuesday. On Thursday he had 7 less than Wednesday and this was 3 times as many as average. How many did he have on Thursday?

12.1 (a) If $f = d \times 4 + 2$, find a formula for d.

(b) Change the subject of $w = \frac{4b - 1}{7}$

(c) The volume of a square-based pyramid with side length s and height h is $V = d^2h \div 3$. Find a formula for h.

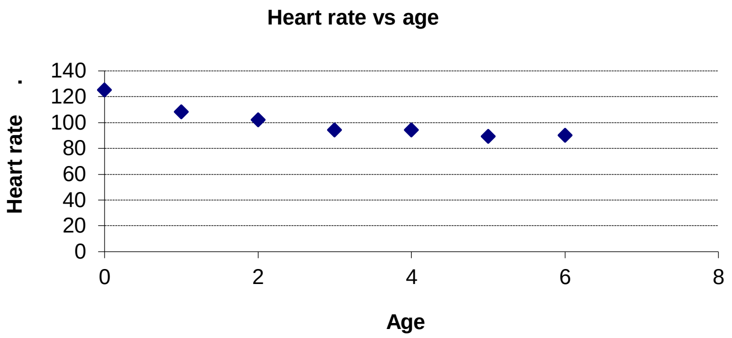
(d) The volume of a square-based pyramid of side length s and height h is $V = d^2h \div 3$. Find a formula for d.

Answers

1.1 (a) 94 (b) 1
1.2 (a) 49 years (b) 40
1.3 (a) 23° (b) 11 a.m.
2.1 (b) Independent – age Dependent – heart rate
(b)

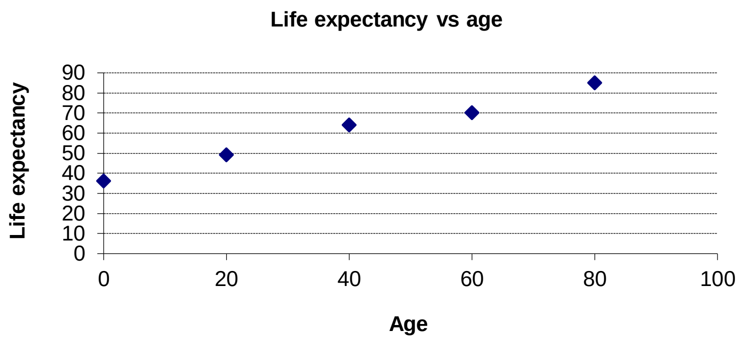
Age (years)	0	1	2	3	4	5	6
Heart rate (beats/s)	125	108	102	94	94	89	90

(c)



2.2 (a) Independent – age Dependent – life expectancy
(b) (0, 36), (20, 49), (40, 64), (60, 70), (80, 85) where the first number is the age in years and the second is the life expectancy in years

(c)



2.3 (a) Independent – time Dependent – temperature
(b) (7, 18), (8, 21), (9, 23), (10, 26), (11, 27), (12, 28), (13, 28), (14, 29), (15, 27), (16, 26), (17, 24), (18, 24) where the first number is the number of hours since midnight and the second number is the temperature.

(c)

Time	7am	8	9	10	11	12	1	2	3	4	5	6pm
Temperature	18	21	23	26	27	28	28	29	27	26	24	24

3.1 (a) No (b) Yes (8, 15.5), (9, 17) (c) Yes (14, 140), (15, 75) approx
(d) No (e) As a formula

4.1 (a) Discrete (b) Discrete

4.2 The values in between are not part of the relation and so should not be plotted.

5.1 $3900

5.2 140

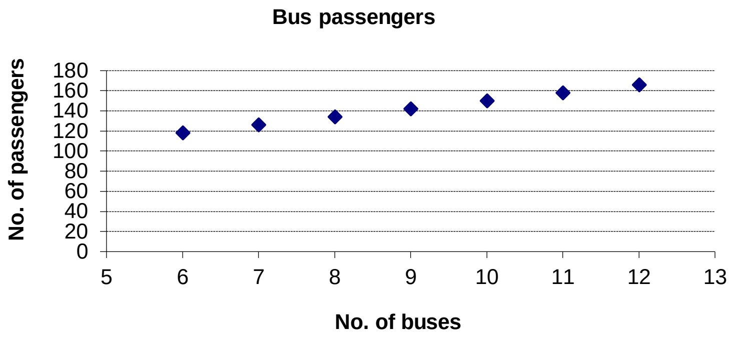
6.1 (a) *fare* = $3.50 + *distance* × $4

(b) *fee* = \$25 + *length of visit* × \$30

(c) *bus fare* = 90 + *distance* × 0.5

6.2

Number of buses	6	7	8	9	10	11	12
Number of passengers	118	126	134	142	150	158	166



7.1 (a) yes (b) no (c) yes (d) no

8.1 (a) \$180 (b) 600

9.1 (a) 18 (b) 48 (c) @@ (d) 4.9

9.2 (a) 14 (b) 11 (c) 25.8

9.3 (a) 6 (b) 23 (c) 8.5 (d) 11 (e) $84^1/_3$

10.1 (a) 22 (b) \$33.50 (c) 210 (d) 48

11.1 (a) 32 (b) \$9.50 (c) 19

11.2 (a) 4 (b) 12 (c) 13

11.3 (a) 4 (b) 3.2 (c) −1.64 (d) 4 (e) −1 (f) \$52

11.4 (a) \$47.60 (b) 9

12.1 (a) $d = \dfrac{f-2}{4}$ (b) $b = \dfrac{7w+1}{4}$ (c) $h = \dfrac{3V}{d^2}$ (d)

$d = \sqrt{\dfrac{3V}{h}}$