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Deriving the Quadratic Formula - 1

If $a \neq 0$ and $ax^2 + bx + c = 0$ then

$4a^2x^2 + 4abx + 4ac = 0$ {multiply both sides by $4a$}

$4a^2x^2 + 4abx = -4ac$ {subtract $4ac$ from both sides}

$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$ {add b^2 to both sides}

$(2ax)^2 + 4abx + b^2 = b^2 - 4ac$ {since $4a^2x^2 = (2ax)^2$}

$(2ax + b)^2 = b^2 - 4ac$ {factorise the perfect square}

$2ax + b = \pm\sqrt{b^2 - 4ac}$ {take square root of both sides}

$2ax = -b \pm\sqrt{b^2 - 4ac}$ {subtract b from both side}

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ {divide both sides by $2a$}

Deriving the Quadratic Formula - 2

Solve for x: $ax^2 + bx + c = 0$

Can you justify each line of the derivation of the formula?

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$a(x^2 + \frac{b}{a}x) = -c$$

$$a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) = -c + a \cdot (\frac{b}{2a})^2$$

$$a(x + \frac{b}{2a})^2 = -c + \frac{ab^2}{4a^2}$$

$$a(x + \frac{b}{2a})^2 = -c + \frac{b^2}{4a}$$

$$a(x + \frac{b}{2a})^2 = \frac{-4ac + b^2}{4a}$$

$$(x + \frac{b}{2a})^2 = \frac{-4ac + b^2}{4a^2}$$

$$\sqrt{(x + \frac{b}{2a})^2} = \pm\sqrt{\frac{-4ac + b^2}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Deriving the Quadratic Formula - 3

Let $$x = z - \frac{b}{2a}$$

Then $$ax^2 + bx + c = 0$$

$$a\left(z - \frac{b}{2a}\right)^2 + b\left(z - \frac{b}{2a}\right) + c = 0$$

$$a\left(z^2 + \frac{2bz}{2a} + \frac{b^2}{4a^2}\right) + b\left(z - \frac{b}{2a}\right) + c = 0$$

$$az^2 + \frac{2abz}{2a} + \frac{ab^2}{4a^2} \quad + \quad bz - \frac{b^2}{2a} + c = 0$$

$$az^2 - bz + \frac{b^2}{4a} \quad + \quad bz - \frac{b^2}{2a} + c = 0$$

$$az^2 + \frac{b^2}{4a} - \frac{b^2}{2a} + c = 0$$

$$az^2 + \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a} = 0$$

$$az^2 + \frac{b^2 - 2b^2 + 4ac}{4a} = 0$$

$$az^2 + \frac{-b^2 + 4ac}{4a} = 0$$

$$az^2 = \frac{b^2 - 4ac}{4a}$$

$$z^2 = \frac{b^2 - 4ac}{4a^2}$$

$$z = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$z = \frac{\pm\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$z = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$z = x + \frac{b}{2a}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$$

Deriving the Quadratic Formula – 4

I'm working on an Intermediate Algebra text which has the usual complete the square section. As has been noted by others, knowing how to complete the square is a very useful skill. However, in the next section, I use the following derivation (which avoids a lot of fractions) for the proof of the quadratic formula theorem:

If

a <> 0 and

ax^2 + bx + c = 0

then

4(a^2)x^2 + 4abx + 4ac = 0

4(a^2)x^2 + 4abx = -4ac

4(a^2)x^2 + 4abx + b^2 = b^2 - 4ac

(2ax + b)^2 = b^2 - 4ac

2ax + b = +-sqrt(b^2 - 4ac)

2ax = -b +- sqrt(b^2 - 4ac)

x = [-b +- sqrt(b^2 - 4ac)]/(2a)

You may notice that the proof is a lot less cluttered than the formal complete the square process. Because it is a derivation that requires only factoring skills, multiplying by 4a and mysteriously adding b^2 to both sides can be justified on the basis that they produce an expression whose factors can be guessed AND checked. The third step from the last is based on a usual Lema [if x^2 = k, then x = +-sqrt(k)] to avoid factoring an atrocious looking difference of squares. Would this be more difficult for a student to reproduce than the usual complete the square process? I don't know! But most students bog down in the bad rational arithmetic even if they have complete the square in their pocket.