Index laws Sequence

While the main focus of this chapter is the index laws themselves, a possibly more important focus is the logical development and proof of the index laws. Equally important is the extension of the index laws to all real numbers. To do so, we have to redefine what is meant by an index, while retaining existing results.

Indices and exponents as a shorthand notation for repeated multiplication

$a.a.a = a^3$

$a.a.a.b.b.b.b = a^3b^4$

Powers of 10
$10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$

Scientific notation for large numbers

Investigation

$3 \times 10\,000 = 30\,000$
$3 \times 10\,000 = 3 \times 10^4$
$\therefore \quad 30\,000 = 3 \times 10^4$

$2.5 \times 100\,000\,000 = 250\,000\,000$
$2.5 \times 100\,000\,000 = 2.5 \times 10^8$
$\therefore \quad 250\,000\,000 = 2.5 \times 10^8$

$3.47 \times 10\,000\,000\,000\,000 = 34\,700\,000\,000\,000$
$3.47 \times 10\,000\,000\,000\,000 = 3.47 \times 10^{13}$
$\therefore \quad 34\,700\,000\,000\,000 = 3.47 \times 10^{13}$

In each case we have written the number in the form: a number between 1 and 10 multiplied by a power of 10.

Reason for sci not: it is
- more compact
- easier to grasp the size of large numbers when written in sci not

Converting from a number to sci not
Converting from sci not to a number
Lots of interesting applications
- see Jacobs
- number of possible Powerball tickets
- astronomical information

Can we perform operations – e.g. add, subtract, multiply, divide – on indices? We can, because indices are just numbers.

Index law 1 - Multiplying powers

Investigation

$a^3 \times a^4 = a \times a \times a \quad \times \quad a \times a \times a \times a = a^7$, etc

Discover rule

Simplify: $a^3 \times b^2 \times a^4 \times b^6$, $3a^2 \times 2b^6 \times a^5 \times 4b^9$, etc

Application: Multiplying numbers in sci not

Index Law 2 - Dividing Powers

Investigation

$a^7 / a^3 = a.a.a.a.a.a.a / a.a.a = a.a.a.a = a^4$

Discover rule

Simplify a^4b^5 / a^2b^3, etc

Application: dividing numbers in sci not

Sci not and a calculator

Expressing numbers in sci not

Operations with sci not

Lots of interesting applications

Investigation - Engineering format

All powers are powers of 3

Converting from sci not to engineering format and vice versa

Index Law 3 - Raising a power to a power

Investigation

$(a^3)^2 = a.a.a \times a.a.a = a^6$, etc

Discover rule

Index Law 4 - Power of a product equals the product of the powers

Investigation

$(ab)^3 = ab . ab . ab = a.b.a.b.a.b = a.a.a.b.b.b = a^3b^3$, etc

Discover rule

Write w/o brackets $(2a)^4$, $(3a^2b^3)^2$, etc

The Big Step – Extending index laws to indices that are not natural numbers

Some history – the reluctance of allowing this

Conditions – however we extend our index laws, they must not contradict existing index laws.

Index Law 5 – the index 0

Investigation

Discussion - what is a sensible interpretation of 5^0?
Check with a calculator. Raise other numbers to the power 0. Make a conjecture.
Now prove conjecture

Now $5^4/5^4 = 5^{4-4} = 5^0$
And $5^4/5^4 = 1$
$\therefore$ $5^0 = 1$

We have a premise: $a^0 = 1$
Test with a calculator.

Class discussion

But what about 0^0?

Sasha says, "Easy-smeasy. 0 to any power is 0."
Kirby says, "Easy-smeasy. Any number to the power 0 is 1"
Who is right?

Some discussion about the big step we have taken here. We have extended our definition of indices. Why would be want to do this? Discuss reasons, esp because we can, extending sci not, and a brief chat about the immense importance of calculus.

Index Law 6 – negative indices

Investigation

Discussion - what is a sensible interpretation of 10^{-1}?
Check with a graphics calculator. Try other negative powers. Make a conjecture.
Now prove conjecture

Standard examples and exercises, inc write with pos indices: $3a^{-4}$.

Application – sci not for very small numbers

Similar development as for sci not for very large numbers.
Lots of interesting applications

Index Law 7 – unit fraction indices

Investigation

Discussion – what is a sensible interpretation of $4^{\frac{1}{2}}$?
Check with a graphics calculator. Raise other numbers to the power ½. Make a conjecture.
Now prove the conjecture:

From Index Law 3: $(4^{\frac{1}{2}})^2 = 4^{\frac{1}{2}\times 2} = 4^1 = 4$

But also $(\sqrt{4})^2 = 4$ {inverse operations}

$\therefore$ $4^{\frac{1}{2}} = \sqrt{4}$

Extend this result to other unit fractions.
State the result.

Exercises

Include some that use both negative indices and unit fraction indices

Index Law 8 – fraction indices

Investigation

What is a sensible interpretation of $8^{\frac{2}{3}}$? $8^{\frac{4}{3}}$? $9^{\frac{3}{2}}$?
Make a conjecture.
Now prove the conjecture, as follows:

$8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = 2^2 = 4$, etc

Writing a fractional power as a root.
Writing a root as a fractional power.

When simplifying by hand, it is easier to root first and power later, as the numbers stay small. Power first and root second is still correct, but often difficult without a calculator.

Exercises

Include some that use both negative indices and fraction indices

Applications

Kepler's Law
Find some other interesting applications

Finally extend index laws to all real numbers.

(I don't believe this can be easily proven.)

Summarise all index laws.

Lots of exercises to consolidate.