M1Maths

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Skills – Mathematical Modelling

- understand the process of mathematical modelling
- produce a mathematical model of a real-life situation, making and documenting reasonable assumptions and values for variables
- process the maths to produce applicable results
- apply the mathematical results to the real-life situation to generate useful information
- assess the validity and reliability of the information generated

Introduction Worked Examples Tasks for Students Internet Resources

Introduction

What is mathematical modelling?

Mathematical modelling is taking a real-life situation, representing the important aspects of it mathematically (as a mathematical model of the situation), doing the mathematics, applying the results to the real-life situation and hence drawing some useful conclusions or predictions about the situation.

Modelling is often associated with problem solving, but it differs in the following way.

In problem solving, the problem is generally presented with all the information required (and usually no unneeded information) to get an exact solution, and there is a single right answer.

In modelling, there is an issue about which some information is available but about which we wish to know more. The information may not be complete or exact, so some assumptions and approximations generally need to be made. This means that results will not be exact and that different approaches can lead to significantly different results (though they should all be in the same ballpark).

A situation can be modelled in different ways, some requiring only basic mathematical knowledge like what a Year 7 student might have, others requiring more sophisticated techniques that might not be possessed until the end of high school or later.

Why learn mathematical modelling?

In many situations in life where mathematics needs to be applied, all the required information is present in exact form. For instance, if we know we will borrow $8000 at 7.5% pa reducing interest and that repayments are to be made monthly for 4 years, we can work out exactly what the repayments will be.

However, there are also many situations where we need to use some mathematics to find out something we need to know, but where the required input information is not all available in exact form. For those situation, mathematical modelling is necessary.

We can guess and estimate what is unknown and do the maths with that data and people are generally capable of doing that. This is in fact mathematical modelling done in an informal way. Formal mathematical modelling makes the guesses and estimated more explicit so that they can be scrutinised and maybe improved, and it includes an assessment of the results as they apply to the real-life situation.

Learning about mathematical modelling can make the process more transparent and reliable. This is particularly important if the mathematics is being done for someone else who will require an indication of its reliability.

Spelling

In UK English (and the English used in Australia and New Zealand and many other British Commonwealth countries) modelling is spelt with a double L – modelling. In US and Canadian English (as well as English as used in many other parts of the world), it is spelt with a single L – modeling. As this is an Australian site, the double L is used.

Organisation of the Module

Following this introduction are three examples of modelling tasks, each worked through at various levels that students might work at.

Then there is a list of suggestions for tasks that can be assigned to students. In some cases, some notes on possible approaches are included, though none have full treatments.

Finally, there is a list of other modelling resources available on the Internet.

Worked Examples

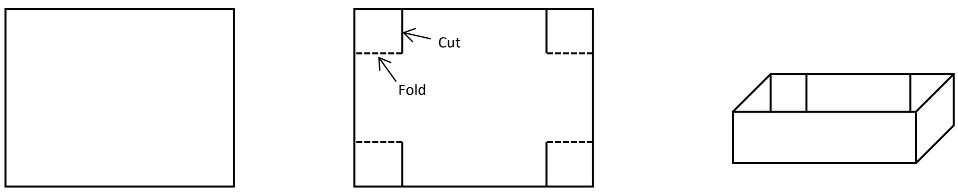
Popcorn box

The situation

A school club is holding a movie night. They are going to sell popcorn. It will be sold in boxes, each box made from a single A4 sheet of card and some glue. How should they make the box to fit the most popcorn?

The treatment

The boxes probably don't need a lid as this would make it harder to eat the popcorn. One way to make an open box is to cut a square from each corner of the sheet of card, then to fold up the sides. If the square is left attached along one edge, then it can act as a tab for gluing.



It will be assumed that the boxes are made this way.

The amount popcorn the boxes will hold can be modelled by the volume of the box, which is given by

$$Volume = length \times width \times depth$$

A4 paper is 29.7 cm long by 20.9 cm wide. The one variable in construction is the size of the square tab in each corner. Let's call this s.

The length of the box is then $29.7 - 2s$; the width is $20.9 - 2s$; and the depth is s.

We then have

$$Volume = (29.7 - 2s)(20.9 - 2s)s$$

We now have to find what value of s gives the largest volume. We can substitute a few values and record the volumes.

s	*Volume*
1	524
2	869
3	1059
4	1120
5	1074

6	945
7	758
8	537
9	305
10	87

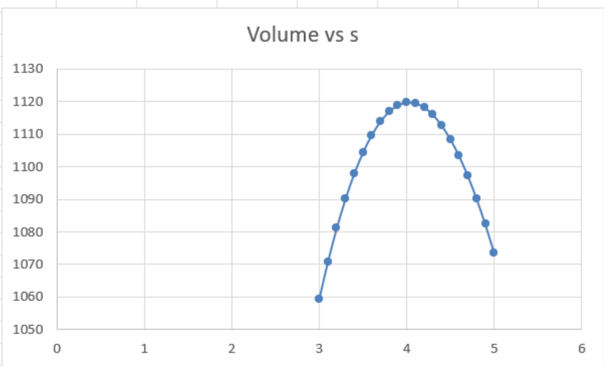
Clearly the volume increases as s increases up to a point, then starts to go down again. It seems to be highest around *s* = 4. We can repeat the calculations in smaller steps between say *s* = 3 and *s* = 5. We might use a spreadsheet for this.

=(29.7-2*B2)*(20.9-2*B2)*B2

B	C	D
s	V	
3	1059.39	
3.1	1070.895	
3.2	1081.12	
3.3	1090.089	
3.4	1097.826	
3.5	1104.355	
3.6	1109.7	
3.7	1113.885	
3.8	1116.934	
3.9	1118.871	
4	1119.72	
4.1	1119.505	
4.2	1118.25	
4.3	1115.979	
4.4	1112.716	
4.5	1108.485	
4.6	1103.31	
4.7	1097.215	
4.8	1090.224	
4.9	1082.361	
5	1073.65	

This tells us that the best value is indeed very close to 4.0 cm.

Another way to see this is with a graph. The one below is generated by the spreadsheet. It confirms that the optimum value for *s* is very close to 4 cm.



Thus it is concluded that the boxes should be made as above using corner tab squares of 4 cm. This gives a box volume of 1120 cm^3.

Evaluation of result

If we assume that the boxes will be made as described above, it does seem that a 4 cm depth is best.

However, this assumes that the popcorn will fill the box to the level of the tops of the sides, but that it will not be piled up above that level. Popcorn can be piled up to some extent, though we wouldn't want it to be piled up to the extent that it spills.

Modest piling up would increase the amount of popcorn that the box would hold. Would the optimum box still be 4 cm deep? Making the value of s slightly smaller won't change the volume much, but will make the box longer and wider, allowing more popcorn to be piled up without risk of spilling.

Thus, if the popcorn is going to be piled up, a slightly smaller value of s might be optimal. As any piling up would be hard to quantify, all we can really say is that a value of s a little below 4 cm might be optimal. 3.6 cm might be a suitable value. We can see from the graph that this would reduce the volume of the box by just 10 cm^3 while allowing a bit more piling up.

Notes on the process

In the process above an assumption was made that the box would be made a certain way. The volume of the box was then used as a model for how much popcorn the box would hold. Of course this is not exactly the same as the amount held, but it is a reasonable model. The variable s was defined and then two formulae were used as models for volume (and thus the amount of popcorn). These were then converted to tables produced by hand and by a spreadsheet and a graph – further models of the situation. A conclusion was then drawn. Finally thus was evaluated in light of the original situation and a possible improvement was suggested.

Overall, this treatment followed the process of mathematical modelling in that a real-life situation was modelled by various mathematical representations, the mathematics was then processed, and finally the result was interpreted in terms of the real-life situation, giving a suggested plan of action.

Epidemic

The situation

The Covid epidemic of 2020-2022 required authorities to make predictions about the spread of the disease, about demands on hospital facilities, about dealing with the dead, about the effectiveness of different measures designed to prevent the spread etc.

None of these predictions could be made exactly or totally reliably, as the starting information (the variables – like how readily the disease spreads, how well people would follow social distancing requirements etc.) weren't known exactly. All we could do was make a mathematical model of the situation using best estimates for the values of these variables, and then process the model to make the required predictions.

Such a model is a simplification of the real situation (which, in this case, was very complex with many variables and unknowns). Although the models were not exact, they generally provided more information and better guidance than simple guesswork without modelling.

Let's assume we start with the following information for the island state of Makura (population 520 000) and would like to make some predictions about how the disease will evolve.

Week	1	2	3	4	5
Cases reported	2	15	67	221	690
Hospitalisations	0	0	4	14	63
Deaths	0	0	1	3	12

Let's assume also that it is known from monitoring a few patients that from the time of picking up the virus to becoming sick is generally about 1 week and that the person is then sick and infectious for a week after which they recover (and are then immune) or die. It also seems that on average, each person who gets the disease passes it on to 2 others.

We need a mathematical model of the disease spread that can be extended beyond Week 5.

How is this affected if we can bring down the number infected per sufferer?

The treatment

Moon Rocket

The situation

We wish to fire a rocket to hit the Moon.

The treatment

We will make some assumptions:

1. Earth is the frame of reference, so we will consider the earth to be stationary (though rotating once per day)
2. Newtons law of gravity applies and g at the Earth's surface is 9.8 m/s^2

3. The radius of the Earth is 6370 km, it rotates once per day, the Moon is 400 000 km from the Earth and orbits it every 28 days
4. The rocket is accelerated at the start of the journey and its motion after reaching 7000 km from the centre of the Earth is affected only by the gravity of the Earth
5. All motion is in the plane in which the moon orbits the Earth
6. The rocket is launched from near the equator where the Earth's rotational velocity is greatest
7. x and y coordinates are defined in that plane with the centre of the Earth at the origin
8. The view is from above the North Pole, so the Earth rotates and the Moon orbits anti-clockwise

We will take the following as variables

- Time, t (in hours)
- The x and y coordinates of the rocket (in megametres)
- The components of the rocket's velocity, x' and y', parallel to the x and y axes when the engines cut out (in megametres per hour)
- The components of the rocket's acceleration, x'' and y'', parallel to the x and y axes at any time after the engines cut out and before it hits the moon (in megametres per hour per hour)

A calculus treatment of this problem would be beyond high-school maths, so a numerical approach with a spreadsheet will be used. We will consider the position, velocity and acceleration of the rocket at regular intervals after the engines cut out. We will try intervals of 0.1 hours initially.
We can set up a spreadsheet as shown below. This spreadsheet and the formulae in it will be out mathematical model of the situation.

Tasks for Students

Space inside a cardboard box
Tiles needed to tile a floor
How many small cubes to make a larger cube
Litres of water in an oval-cylindrical water truck tank
Windscreen wipers
Running a restaurant
Borrowing to invest
Drought declaration index
City liveability
Fox and rabbit populations
Vitamin D from the sun
Clustering of towns
Water in a hemispherical container
Hyperthermia
Late heavy bombardment
Cost vs volume for box with double flaps top and bottom
Cost vs volume for cylindrical water tanks
Max area for fenced enclosure against a wall
Other optimisation tasks
Flat-rate or reducing loan
Telling if probability data is real or made up
Sensitive question surveys
If pokies pay out 85%, how likely are you to come out ahead after a session?
The cost of smoking/vaping
Markov chain problem
Water in a partly filled spherical container
Chaotic fish population evolution
Finding formulae by guess and check / regression
Prisoner release games, e.g. with dice differences
Greedy pig
Collecting complete sets of items gained at random
Benford's law
Probability that a trend in data is real
Relation between slope angle and gradient
Lengths of diagonals of rectangles with known dimensions (by drawing)
Volumes of pyramids
Number of squares in grids of squares
Astronomical distances (leading to scientific notation)
Permutation and combination situations
Spread of favourable mutations

Internet Resources

Mathematical models
https://www.mathsisfun.com/algebra/mathematical-models.html
A good introduction to simple modelling with several worked examples

IM²C: International Mathematical Modeling Challenge
https://www.immchallenge.org.au/supporting-resources/what-is-mathematical-modelling
A comprehensive introduction with examples

Mathematical modelling in the junior secondary years: An approach incorporating mathematical technology
https://aamt.edu.au/wp-content/uploads/2020/10/4-amt74_1_lowe.pdf
An introduction to the theory of mathematical modelling