M1 Maths

Investigating

- investigate using the approach: data, pattern, show, extend

Introduction Level 1 Level 2 Level 3 Level 4 Level 5 Level 6

Introduction

What is a mathematical investigation?

A mathematical investigation is like a problem except that there is no specific question to be answered. Rather, there is a stimulus which you use to explore some mathematical ideas. The idea is to find out as much as you can about the situation. Different people doing the same investigation might explore and find out quite different things.

Investigational work has one important characteristic in common with problem solving – they both involve searching for new information related to that given. Investigating can thus assist you to develop your problem solving skills.

Investigating has another benefit, however. It helps get you into a way of thinking where you can find things out for yourself, discover your own mathematics and realise that you can do things in mathematics that you haven't been shown how to do by someone else.

It can also be interesting.

In doing an investigation you are working the way pure mathematicians do, discovering new mathematical facts and ideas. Investigations allow you to show and develop your flair and creativity.

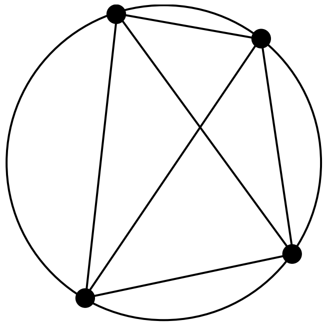
In doing most investigations you will go through four stages:

- Collect data and organise it systematically
- Find patterns and make and test conjectures
- Show that your conjectures must be true
- Extend the investigation

Here is a description of how an investigation might proceed. This stimulus is:

Dots are placed on the perimeter of a circle and are joined by lines. Investigate.

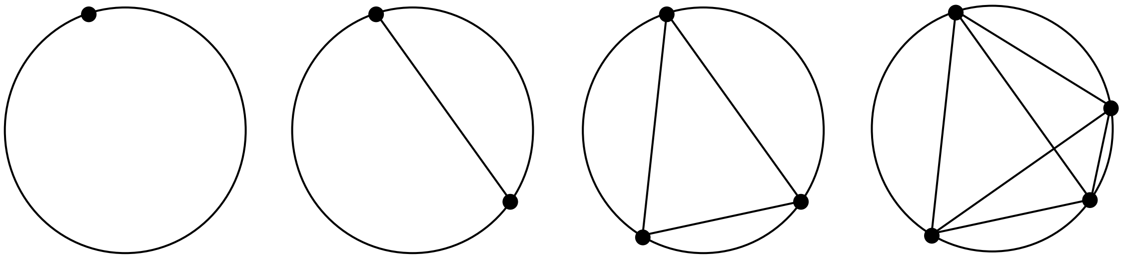
There is no set question to answer here. There are various ways we could go. To give us some ideas, let's draw a circle, put some dots on the perimeter and join them with lines.



One thing that might be worth investigating is the number of lines for different numbers of dots. In the diagram we have 4 dots and six lines. We could find out how many lines for 1 dot, 2 dots, 3 dots etc.

Collecting data and organising it systematically

We would sketch the drawings, collect the data and organise it systematically into a table.



Number of dots	1	2	3	4	5	6	7	8	9
Number of lines	0	1	3	6					

Finding patterns and making and testing conjectures

An investigation is about finding out as much as possible. We have found how many lines for 0, 1, 2, 3, and 4 dots. But there are lots of other numbers. We could go on drawing more pictures with more and more dots, but it will get harder and harder to draw and count all the lines.

A better way is to try to find a pattern in the numbers and use that to predict the numbers of lines for other numbers of dots, like 70. We can't really find a pattern from two bits of data, so we keep collecting until we do have enough.

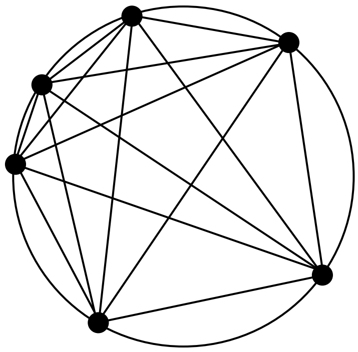
Using out 4 data points above, there seems to be a pattern that the number of lines goes up by 1, then by 2, then by 3. We are not certain that this pattern will continue, but we can make a conjecture that it does. A conjecture is an idea – something that you think might be true, but which you are not sure about. The word *hypothesis* means the same thing. Scientists tend to use the word *hypothesis*, whereas mathematicians use the word *conjecture*. This is maths, so we will talk about conjectures.

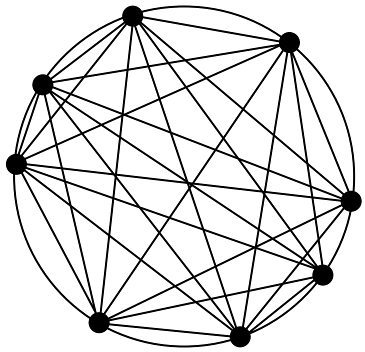
So we make a conjecture that this pattern continues and the number of lines will next go up by 4, then by 5, then by 6 and so on . . . like this:

Number of dots	1	2	3	4	5	6	7	8	9
Number of lines	0	1	3	6	10	15	21	28	36

To put this another way, when we add a new dot, the number of lines increases by the number of dots already there.

To be a bit more sure of our conjecture we test it by trying some other numbers of dots and checking to see if the number of lines is the same as what our conjecture predicts. We might check 6 dots and 8 dots.





The drawings show that our conjecture holds true for 6 and 8 dots. This makes it more likely that it will be true for all numbers of dots, though it is still not certain.

Showing that the conjectures must be true

To be certain that our conjecture is true, we have to prove it logically. This is often the most challenging part of an investigation. It can take a while to find a way to prove a conjecture. The conjecture about the lines can be proved like this.

Say we have 8 dots and 28 lines. When we add a new dot, we will have to draw lines from it to each of the 8 dots that are already there. So the number of lines will increase by 8. Then when we add another dot, we will have to draw lines from it to each of the 9 dots that are already there. So the number of lines will increase by 9. This same logic works whatever the number of dots already there.

Therefore our conjecture is true: each time we add a new dot, the number of lines increases by the number of dots already there; so the number of dots goes up, the number of lines will go up by 1, by 2, by 3 and so on for ever.

Extending the investigation

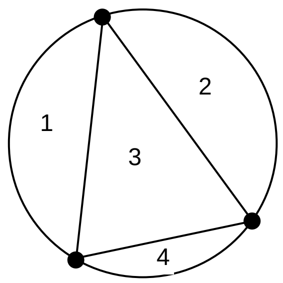
We have made a discovery about the number of lines so that we can predict the number of lines for any number of dots without having to draw the picture. There are various other things we could look at to extend the investigation.

One thing that would be challenging would be to try to get a rule for working out the number of lines from the number of dots without having to work out the numbers of lines for 1, 2, 3, ... dots first.

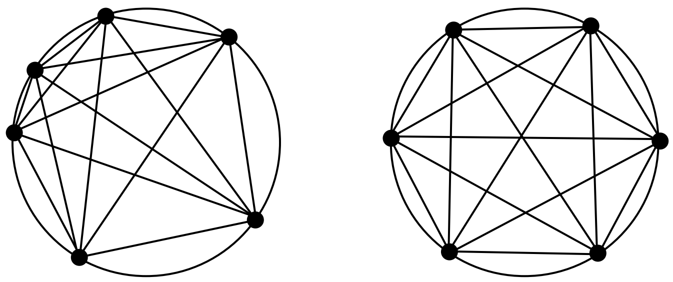
The rule is: *multiply the number of dots by one less than the number of dots, then divide by 2.* So for 10 dots there will be $10 \times 9 \div 2$ lines, i.e. 45 lines.

We can prove this like this: Lines go from every dot to every other dot (i.e. all the dots except one). For example, for 10 dots there will be 9 lines from each of the 10 dots making 10×9 lines. This is 90. But looking at it this way, we will have counted each line twice, so the actual number will be half that. $90 \div 2 = 45$.

There are other ways to extend the investigation. We could look at the number of regions formed by the lines. For example, for 3 dots, there are 4 regions as shown below.



Then we might realise that the number of regions will sometimes be different if the dots are arranged evenly around the circle or if they are arranged randomly.



So we can investigate that too. And so on.

Presenting the Results

You will probably solve quite a lot of problems, but you will do fewer investigations – if only because they tend to take longer.

Unlike with solving problems, an investigation cannot be done in your head. You will need to write down the data as you collect it so you don't forget. In general, you will also write down the patterns you find, the conjectures you make, the results of testing them and you will put your proof of the conjectures in writing.

Basically, you will be recording your results along with a commentary explaining what you decided to do, what you found as a result, and so on.

Sometimes, this will be just for your own eyes and it needn't be done neatly. At other times it maybe for someone else to read – like your teacher – and you will need to be more careful to make sure it is well laid out and easy to read. If you do an investigation as an assessment piece, the mark you get will depend a lot on how well you write.

In most cases, what you write as you go along should be adequate as long as you take care while writing it – you shouldn't need to rewrite it neatly afterwards.

Your work on an investigation might range from a page or two to numerous pages.

The rest of this module consists of a number of investigation that you could do. They are divided into Levels 1 to 6 to give a guide as to when you might try them. However, because when doing an investigation, you investigate whatever you want, you could probably do most of these investigations at any time.

What determines the quality of an investigation is what you do with it, what ideas you have and pursue, how much you discover and how well you explain what you did and found.

Level 1 Investigations

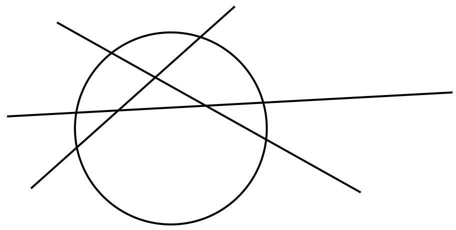
Investigation 1.1: Odds and Evens

Pick two odd numbers and add them. Is the result odd or even? Investigate.

$$35 + 7$$

Investigation 1.2: Cutting a Circle

Make three straight cuts across a circle. How many pieces can the circle be cut into?



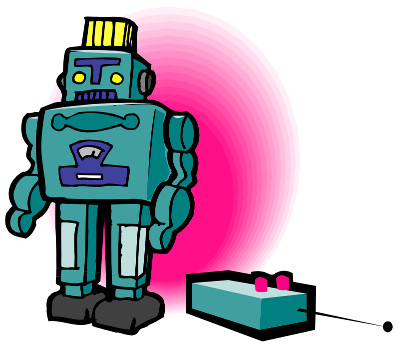
Investigation 1.3: Robot

A robot is programmed to:

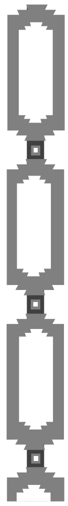
- turn 90° right and walk 1 pace,
- then turn 90° right and walk 2 paces,
- then turn 90° right and walk 3 paces

It then keeps repeating these three steps.

Investigate.



Investigation 1.4: Number Chains



Pick a 2-digit number e.g. 32

Multiply the ones digit by 4, then add the tens digit:

$2 \times 4 = 8$
$8 + 3 = 11$

Take the number you get and do the same to it:

$1 \times 4 = 4$
$4 + 1 = 5$

And again

$5 \times 4 = 20$
$20 + 0 = 20$ and keep going.

Investigation 1.5: Cascades

In a cascade, no number is allowed to have a higher number above it or to the left of it.

Two possible 2 × 2 cascades are:

1	2
3	4

1	3
2	4

1	4
2	3

is not allowed. Can you see why?

Are there any other 2 × 2 cascades?

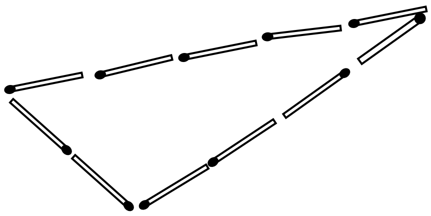
Here is a 2 × 3 cascade

1	2	4
3	5	6

How many others can you make? What about 3 × 3 cascades?

Investigation 1.6: Match Triangles

A triangle is made of whole match sticks. Its longest side is 5 matches long.



Investigation 1.7: Adding

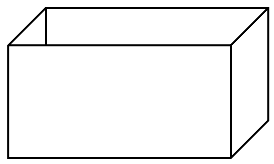
Pick a 3-digit number with no repeated digits.

Find all the 2-digit numbers that can made from the 3 digits.

Add them up.

Investigation 1.8: A4 Container

What is the largest container you can make with an A4 sheet of paper?



Investigation 1.9: Odd Numbers of Factors

The factors of 12 are 1, 2, 3, 4, 6, 12. There are 6 of these; 6 is an even number; so 12 has an even number of factors.

The factors of 16 are 1, 2, 4, 8, 16. There are 5 of these; 5 is an odd number. So 16 has an odd number of factors.

Investigation 1.10: Sums

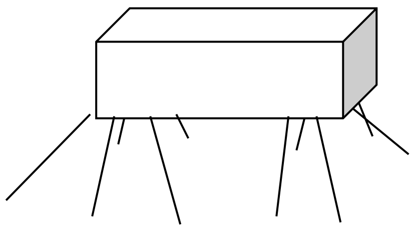
How many ways can you write 5 as a sum of positive whole numbers?

Two possibilities are 1 + 2 + 2 and 4 + 1

Investigation 1.11: Brick

You have a brick, 15 straws and 2 m of sticky tape. You are to build a structure out of the straws that can hold the brick as far as possible off the ground. The measurement will be taken to the lowest point of the brick.

You have 25 minutes, then the judging will take place.

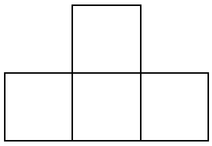


Investigation 1.12: Funny Money

The only money in the Mohelian Republic is \$8 notes and \$5 coins. What amounts of money can be made?

Investigation 1.13: Square Shapes

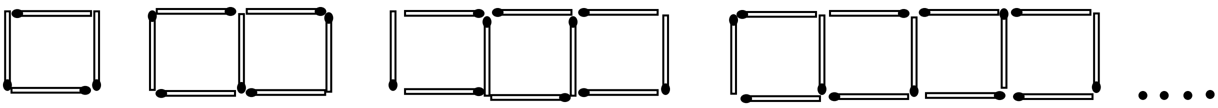
How many different shapes can be made from 4 squares joined along their sides?



Level 2 Investigations

Investigation 2.1: Match Patterns

Make some lines of squares out of matches like this:



Investigate how many matches it takes?

Investigation 2.2: Terminating and Recurring Decimals

$^5/_4 = 1.25$ The decimal places terminate after the first two.

$^6/_7 = 0.85714285714 28$…… The decimal places go on for ever, repeating the same sequence of 6 digits over and over. They recur.

Try other fractions.

Investigation 2.3: One Odd Factor

The number 4 has three factors: 1, 2 and 4. Only one of these is an odd number.
What other numbers have only one odd factor?

Investigation 2.4: Last Digits of Powers

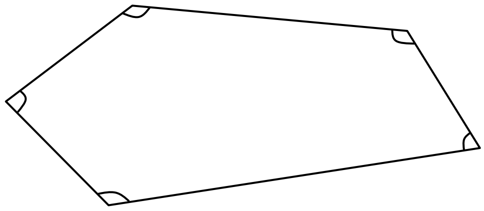
$3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$. The last digit is 3. Investigate the last digit of other powers.

Investigation 2.5: Four Numbers

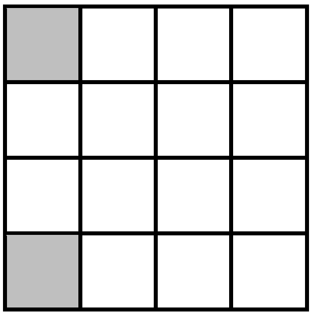
Write down four different whole numbers such that their sum is 12. How many different ways can you do this? Try with a sum of 13 or other numbers.

Investigation 2.6: Angles in a Polygon

Draw a polygon. Measure all the internal angles and add them up. Try this for different polygons. Can you find any patterns? Can you explain the patterns?



Investigation 2.7: Dominoes



On this grid, two squares are blocked out. Show how to divide the rest of the grid into seven 2×1 rectangles.
Try blocking out different pairs of squares.
What would happen if you used a different sized grid, say 5×5 or 6×6 or 5×3?

Investigation 2.8: Coins

Alice hides 3 different denomination coins (say a 5c, a 10c and a 20c) behind her hand. One or two are heads up, the other(s) tails up.

Beetle then gives instructions as follows:

Either 'Turn over the 5c'

or 'Turn over the 10c'

or 'Turn over the 20c'.

When all 3 coins are up the same way (i.e. all heads or all tails), Beetle has won.

What is the best strategy if Beetle wants to be sure of winning in the smallest number of moves.

What if there were more coins?

Investigation 2.9: Magic Squares

In a magic square, all rows, all columns and the two diagonals have to add up to the same number called the magic sum.

8	1	6
3	5	7
4	9	2

In this 3 × 3 square, the magic sum is 15. Add up each row, column and diagonal to check it equals 15.

Can you make other 3 × 3 magic squares?

Can you make magic squares of other sizes?

Can you make magic oblongs (rectangles that aren't squares), e.g. 3 × 5?

Investigation 2.10: CALES - a Game for Two

Lay 13 counters in a row, each one touching the next.

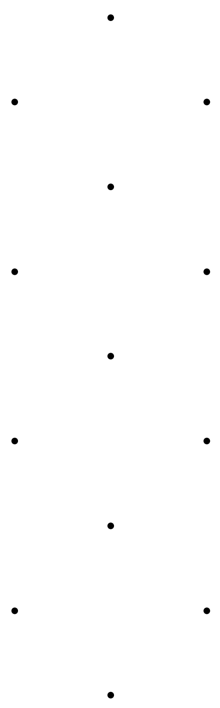
A move consists of removing a counter or two touching counters.

The winner is the person who removes the last counter.

Investigate strategies for winning.

Try variations.

Investigation 2.11: Climbing - a Game for Two



Place a counter on the bottom dot. A move consists of moving the counter either straight upwards to the
the next dot or diagonally upwards to the next dot.

Take turns moving. The winner is the person who moves the counter to the top dot.

Find a way to be sure of winning.

For a change you can make the person who lands on the top dot the loser. Or try other variations.

Level 3 Investigations

Investigation 3.1: Number Columns

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	etc.		

Pick two numbers from this table.

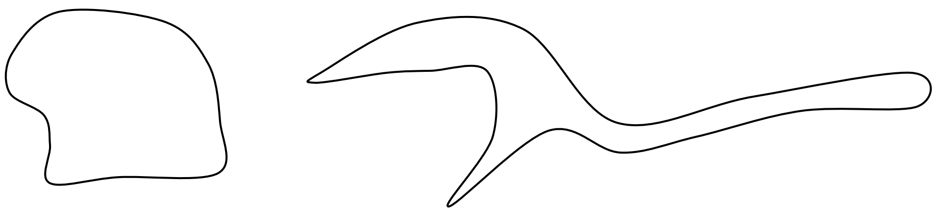
In which column is their sum?

In which column is their difference?

Investigate.

Investigation 3.2: Compactness

The shape on the left is more compact then the shape on the right.



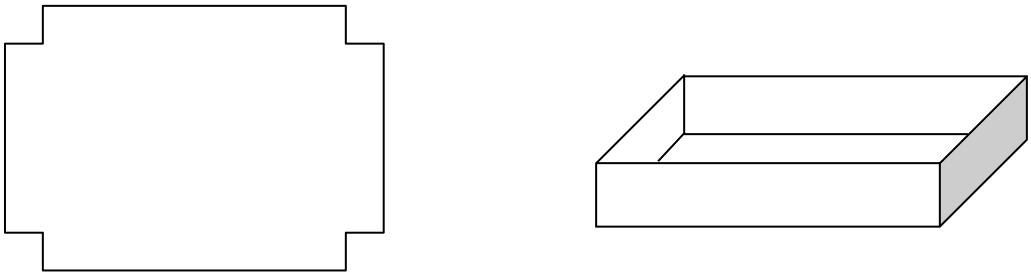
Explore ways of defining compactness so that every shape can be given a number – its compactness index – and more compact shapes have a higher compactness index.

Investigation 3.3: Sums of Cubes

Take a number. Find the cube of each digit. Add up the cubes. This gives you a new number. Do the same to the new number.

Investigation 3.4: Tray

You can make a tray by taking an A4 sheet of paper, cutting a 2 cm square from each corner, then folding the edges up. Like this.



Make one. What is the capacity of the tray?

Make another one by cutting out squares of a different size. What is its capacity?

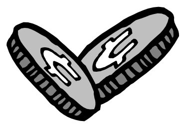
What is the largest capacity tray you can produce? What size squares produce the largest capacity?

HINT: A graph can be very helpful in solving this problem.

Investigation 3.5: Tossing Coins

If you toss a coin a lot of times, it will come up heads about ½ of those times. Another way of saying this is that the probability of getting heads is ½ (or 0.5 or 50%).

Try tossing 2 coins and finding out the probability that both come up heads (in other words the fraction of the times that both come up heads).



Explore other numbers of coins.

Investigation 3.6: Thoan

Think of a number, add 9, multiply by 2, subtract 8, divide by 2, then subtract 5.

Investigation 3.7: Three Numbers

Write down three whole numbers such that their product is 24. How many different ways can you do this?

Investigation 3.8: Unit Fractions

A unit fraction is a fraction with a numerator of 1. $^1/_5$ and $^1/_{72}$ are unit fractions; $^3/_5$ is not.

$^1/_4$ and $^1/_{12}$ are two unit fractions which add to make another unit fraction: $^1/_4 + {^1/_{12}} = {^1/_3}$.

Can you find any other pairs of unit fractions that add to make another unit fraction?

Investigation 3.9: Fraction Threesomes

Try to find a common fraction, a decimal fraction and a percent which add to make 1.

How many different combinations can you find?

Investigation 3.10: Difference

Pick any three digits, e.g. 3, 4 and 8.

Then put them together to make the largest number you can. In this case it would be 843.

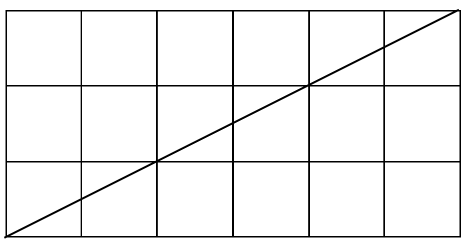
Also, put them together to make the smallest number you can. In this case it is 348.

Then find the difference between the two 3-digit numbers. In this case 843 – 348 = 495.

Investigation 3.11: Diagonals of Rectangles

Print a couple of sheets of the squared paper on the next page.

The rectangle below is made up of squares. The diagonal passes through 6 of the squares.



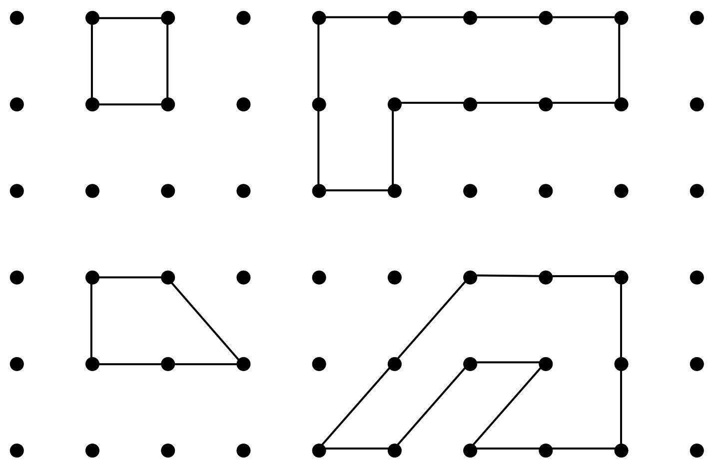
Try other rectangles made up of squares and see how many of the squares the diagonal passes through.

Investigation 3.13: Dotty Shapes

Part 1

You have some dotty paper. The dots are 1 cm apart, so a small square made by 4 dots has an area of 1 square centimetre.

Join the dots to make some shapes with no dots inside.



For each shape, find the number of dots on its perimeter and its area.
Organise the data into a table.

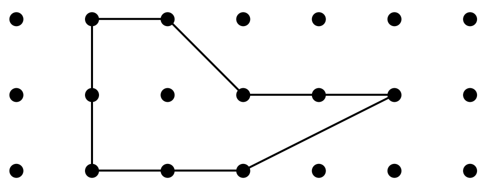
Does the relation between ***number of dots on the perimeter*** and ***area*** have a pattern?

If it does, write it as a formula:

number of dots on the perimeter = area × ......

Part 2

Do the same for shapes with one dot inside.

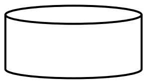


Then for shapes with two dots inside and so on.

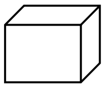
Do you notice an overall pattern. Can you write a formula for it?

Level 4 Investigations

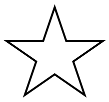
Investigation 4.1: Non-Stop Draw



can be drawn without taking your pen off the paper or going over a line twice.



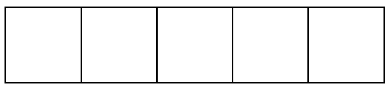
cannot



can

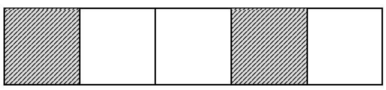
Investigate for other figures and try to find out what it is about the shape that determines whether it can or can't be done.

Investigation 4.2: $^2/_5$ of a Rectangle

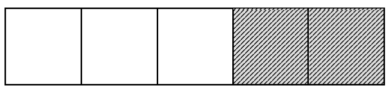


This rectangle is made from 5 squares. You have to completely shade in 2 of those squares.

One way to do this is like this.

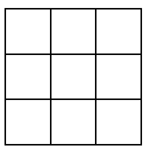


Another way is like this.



How many ways can you find?

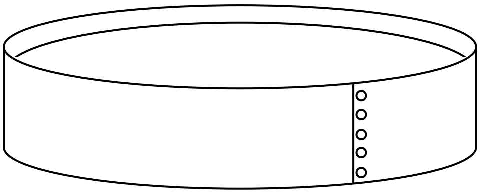
Investigation 4.3: Squares



How many squares here?

Investigation 4.4: Pool

A company makes above ground pools. The pools are made like this:
They take a rectangular sheet of thin metal, bend it into a circle, bolt it together at the join, stand it on flat ground, then put a plastic liner inside and fill it with water to 15 cm from the top.

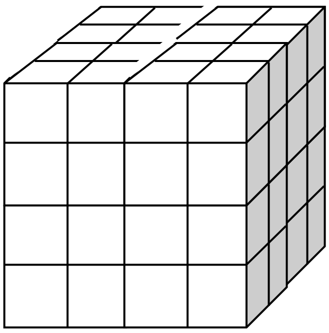


The metal sheets can be any shape of rectangle, but they can only have an area of 12 m^2. So, for example, they could be 12 m by 1 m or 8 m by 1.5 m or 6.4 m by 1.875 m.

Investigate the amount of water the different pools would hold.

Investigation 4.5: Painted Cubes

A cube is made from small MAB blocks glued together.



It is then dipped in paint. How many of the MAB blocks get paint on
(a) 3 faces (b) 2 faces (c) 1 face (d) 0 faces

Investigation 4.6: Consecutive Numbers

Some numbers can be written as the sum of consecutive whole numbers.

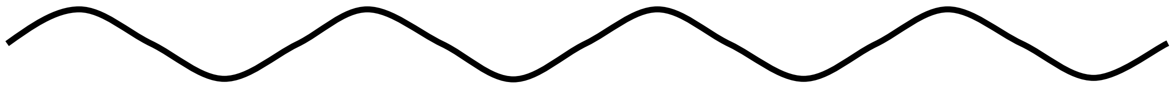
E.g. 9 = 2+3+4 21 = 10+11

Some cannot, e.g. 4

Investigate.

Investigation 4.7: Waves

Investigate the graphs of the function family $y = a \sin b(x + c) + d$



Investigation 4.8: Pseudo-Fibonacci

4	5		9	14	23
7	*b*				114

What number should go in Square *b*?

Level 5 Investigations

Investigation 5.1: Memory

Martin McManus, the Memory Madman, tried to convince his audience that he had an amazing memory. He told them he had memorised the phone book.

He then asked one of them to pick a three-digit number with all the digits different. They picked 863. He wrote it up, then reversed it to get 368. Then he found the difference: 863 – 368 = 495. He then reversed that to get 594, then added 495 and 594 to get 1089.

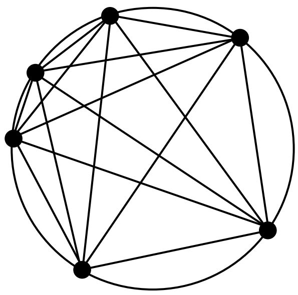
He wrote this number up in two parts: 108-9. He then asked another audience member to turn to page 108 of the phone book and to look at the 9th entry on the page. Without looking, he recited the name, the address and the phone number.

Investigate.

Investigation 5.2: More Dots on a Circle

Put some dots on the edge of a circle and join them with lines.

Investigate the number of white spaces inside the circle.



Investigation 5.3: Snowflake

For this task you need two people. Each needs a coloured pen. Choose two colours that go well together. Red and black is a good choice. Pink and orange isn't.

Print the grid on the next page.

Person 1 takes the black pen and colours a square in the centre of the grid. Then they write the number of squares they coloured (1) in the table below the grid.

Person 2 then takes the red pen and colours every square that shares just one side with the black one. They then write down the number of squares they colour in the table below the grid.

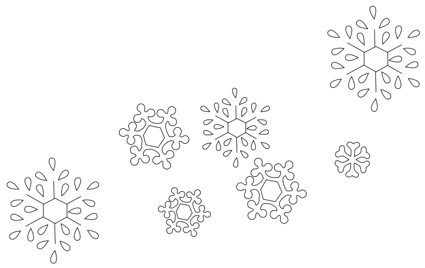
Up to here has already been done for you.

Person 1 then takes the black pen again and colours in every square that shares just one side with the existing picture. They then write down the number of squares they colour. (This hasn't been done for you, but as a check, the number should be 4.)

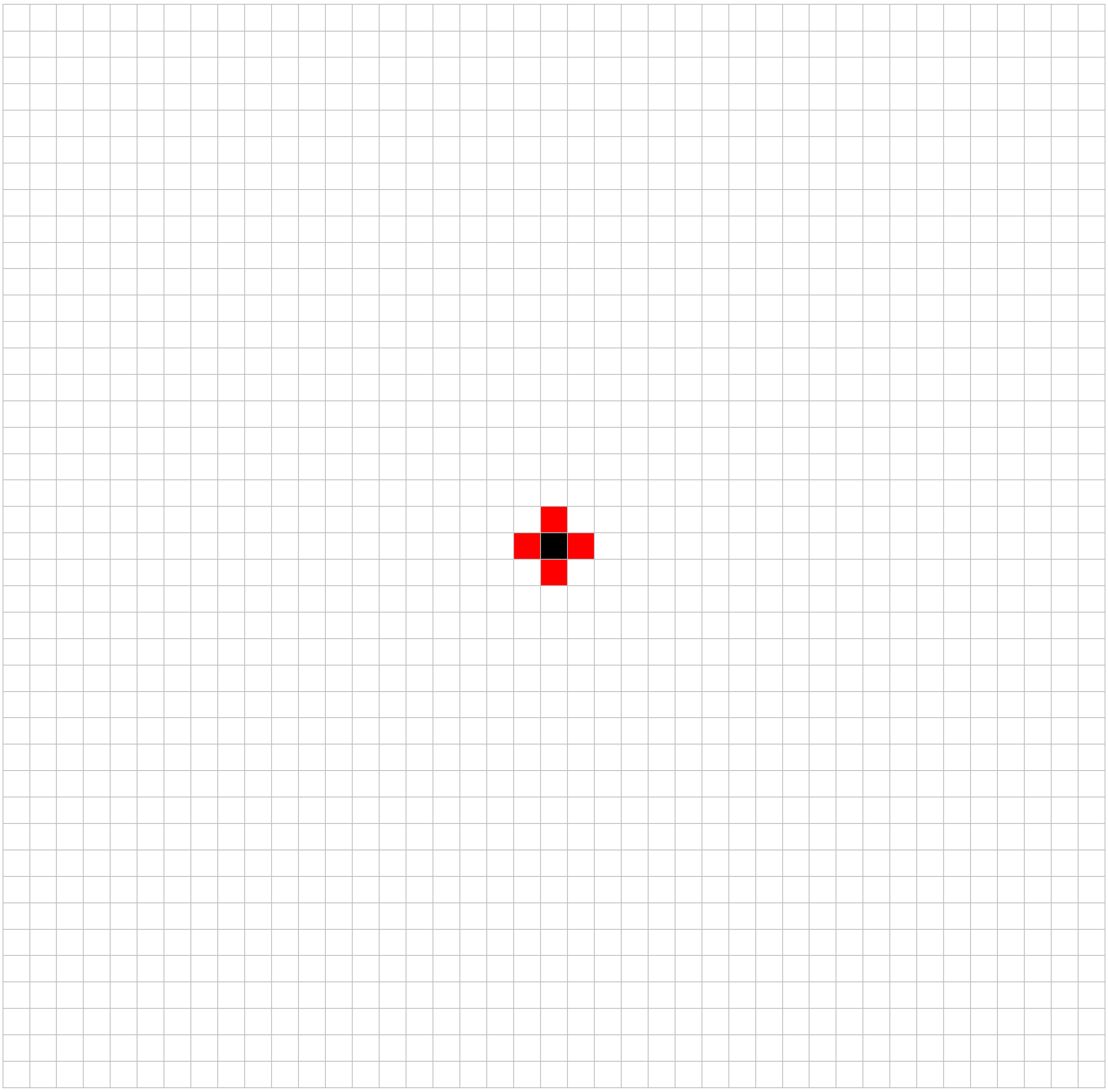
Person 2 (again with the red pen) colours in every square that shares just one side with the existing picture. They then write down the number of squares they colour. (This number should be 12.)

Person 1 (again with the black pen) colours in every square that shares just one side with the existing picture. They then write down the number of squares they colour. (This number should be 4, the next few numbers should be 12, 12, 36, 4 . . .)

Continue like this until you have built up a picture of stunning beauty and a table of numbers along the bottom. Proceed carefully because it is difficult to correct a mistake.



The mathematical part is to investigate patterns in the sequence of numbers so that you can predict the number of squares that will be coloured in without actually doing the colouring.



Go	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Squares coloured	1	4														

Go	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Squares coloured																

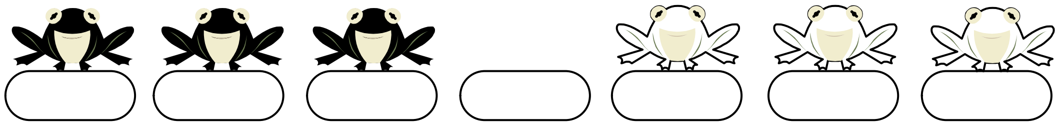
Go	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
Squares coloured																

Go	49	50	51	52	53	54	55	56	57	58	59	60	61	66	63	64
Squares coloured																

Go	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
Squares coloured																

Investigation 5.4: Frogs

Three black frogs and three white frogs are sitting on stones like this.



They have to swap places in as few moves as possible. A move consists of one frog stepping to an unoccupied stone immediately beside him or jumping over one frog immediately beside him onto an unoccupied stone.

Investigation 5.5: $a^b = b^a$

Investigate the equation $a^b = b^a$.

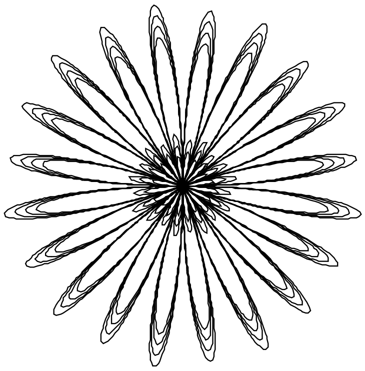
Investigation 5.6: Clustering Index

Consider a map with towns on it. The towns could be randomly distributed, or they could be clustered into groups with bigger spaces between the groups, or, at the other extreme, they could be arranged so that they are all the same distance from their neighbours. Investigate ways to define a clustering index – a single number which indicated the degree of clustering.

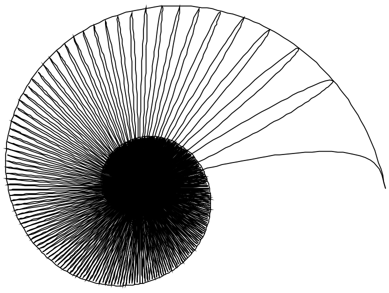
Level 6 Investigations

Investigation 6.1: Polar Graphs

In a polar graph, r, the distance of a point on the graph from the origin is a function of the number of degrees the point is anti-clockwise from the positive x-axis. The number of degrees can go beyond 360.

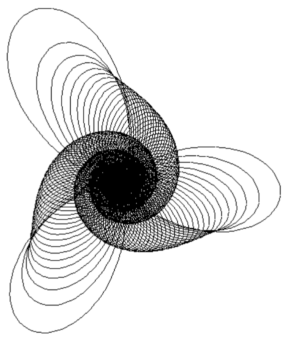
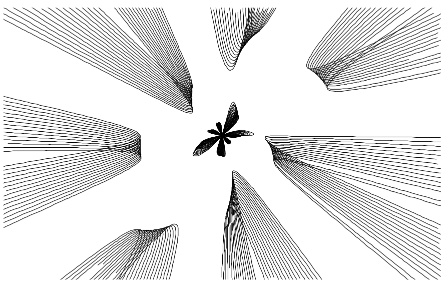
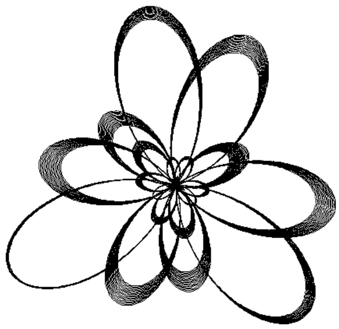
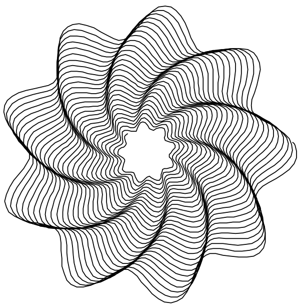
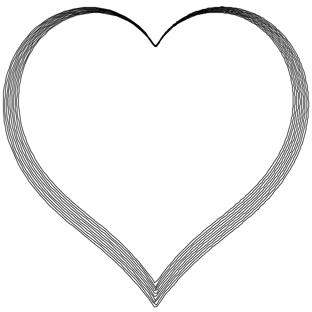
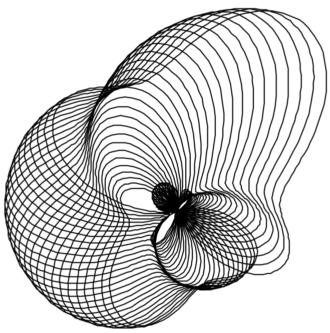


This is the function $r = 0.02\theta \ \sin 10\theta$ over the domain $\theta=16$ to $\theta=32$ and $\theta=81$ to $\theta=106.2$



This is the functions $r = e^{-0.2\theta}$ and $r = \cos^2 (12\theta^2)\ e^{-0.2\theta}$ over the domain $\theta > 0$

Use suitable graphing software (there are plenty of free ones on the Internet) to graph other functions. Try to get pleasing graphs. Here are some you try like to emulate.

Investigation 6.2: x^x

Investigate the relation $y = x^x$

Investigation 6.3: 100% Interest

Euler put $1 000 000 in a savings account getting 100% p.a. compound interest. Investigate how much he would have at the end of the year with differing compounding periods: year, quarter, month, week, day, hour and so on.

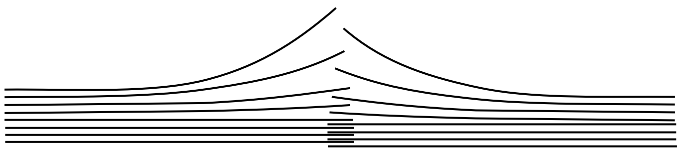
Investigation 6.4: Binomial Expansions

Investigate the expansions of $(x + a)^n$ for different values of n.

You might be able to use your findings to show that the derivative of ax^n is nax^{n-1}.

Investigation 6.5: Fluttering

Fluttering is a shuffling technique in which you split a pack of cards into two equal stacks, take one stack in each hand, then interleave the two stacks.



Few people can do a perfect flutter, but Ernest can. However, you can simulate a perfect flutter by dividing the pack into two, then taking cards alternately from the top and bottom halves to form the shuffled pack.

Ernest took a small number of cards, put them in order A, 2, 3, 4, 5 etc., then fluttered them repeatedly until they were back in the same order. Investigate.