M1 Maths

What is Maths?

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Mathematics is quantitative reasoning.

Reasoning means working out things which follow logically from things we already know. For example, when Sherlock Holmes saw large footprints in the snow leading to the open window, he reasoned that the murderer was big and that he or she came in through that window.

Quantitative means to do with quantities or amounts or sizes – things which require numbers to specify. When Sherlock measured the footprints to be 31 cm long, he might have reasoned that the murderer had size 11 or 12 shoes. When he measured the spacing of the footprints to be 86 cm, he reasoned that the murderer was about 190 cm tall.

In many situations in modern life, quantities are important. If we are making pancakes, mixing some flour, some eggs and some milk may not produce a satisfactory result. Mixing 500 grams of flour, 200 mL of milk and 2 eggs is much more likely to produce good pancakes. In the same way, if you need to paint a room, sending a kid to the hardware store with the instructions *Go buy some cream-coloured paint* might not produce as satisfactory result as working out what you need, then saying *Go buy 4 litres of cream-coloured paint*. And of course, after working for your employer all week, it may not be enough just to get *some money*. There is probably a particular amount that you would be expecting.

Some people argue that geometry is part of maths and is to do with space and shape rather than quantity and number. If you think about the geometry you learn though, the important things are the number of sides, the length of the line (number of centimetres), the angle (how many degrees) and so on. Dealing with shapes without using numbers is more likely to be art than maths.

Mathematics (quantitative reasoning) is coming up with new quantitative information using quantitative information we already have. This is done using common sense, logic and a certain amount of creativity to come up with approaches that will work. Sometimes, it can be hard to find the right approach to solving a problem, so a certain amount of persistence is needed also. To be good at maths, we need to develop our common sense, our logic, our creativity and our persistence.

It is possible to do all maths from first principles using just common sense. However, there are quite a few facts and techniques that we tend to use a lot and it can be very useful to know these rather than to have to work them out every time we need to use them. An example of a fact worth remembering is that $5 \times 9 = 45$. An example of a technique is adding a zero to the end of a whole number to multiply it by 10.

So learning maths is a combination of developing our reasoning, logic, creativity and persistence (we call this Skills) and being able to remember and use certain facts and techniques (we call this Knowledge).

Mental arithmetic is another aspect of mathematical skills. Standard learnt methods (algorithms) aren't always the quickest methods for doing arithmetic mentally: those who are good at mental arithmetic use quite a bit of reasoning, logic and creativity as well as number sense, in doing the arithmetic.

One final skill is the ability to express one's mathematical reasoning and results in a way that is easy to follow – in other words, good mathematical communication.

The Body of Mathematical Knowledge

Pythagoras' Theorem states that the square of the length of the long side (hypotenuse) of a right-angle triangle on a flat surface is equal to the sum of the squares of the other two sides. A lot of other mathematical results are based on this. For instance, the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-dimensional space is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The 4-dimentional result used in relativity is based on this. The space-time interval between two events is given by

$$\sqrt{c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)},$$

where t is time and c is the speed of light.

Now, of course, if it ever turned out that Pythagoras' Theorem was wrong, then these other results would also be wrong – as well as the hundreds or thousands of other results which have been derived using the theorem.

Because of this, mathematicians need to be sure that a result is correct – in all situations – before accepting it. The body of mathematical knowledge is the set of all results that are known to be correct.

Pythagoras theorem is in fact correct. How do we know? We have to prove it. To prove a person guilty in a court of law, we have to show them to be guilty beyond reasonable doubt. To prove a mathematical result we have to show that it is true beyond any doubt whatsoever – we have to be absolutely certain. A mathematical proof is based on logic rather than evidence.

Here is a proof of Pythagoras' theorem.

ABCD and JKLM are squares

$\therefore$ MCL etc. are right-angled triangles

The area of ABCD is $(a + b)^2$

The area of JKLM is the area of ABCD minus the four triangles $= (a + b)^2 - 4\,(\frac{1}{2}ab)$

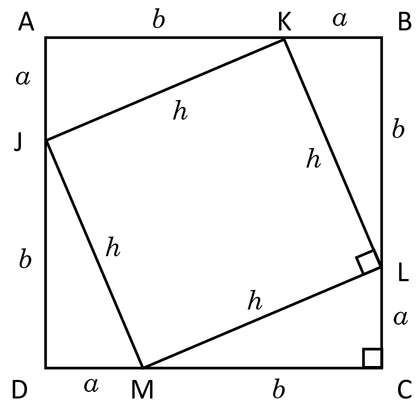
$= a^2 + 2ab + b^2 - 2ab$

$= a^2 + b^2$

But the area of JKLM is also h^2

$\therefore h^2 = a^2 + b^2$

$\therefore$ Pythagoras theorem is true.



But of course, that proof relies on several other statements being true. For instance, we need to assume that the area of a square in side × side, that the area of a triangle is ½ base × height. We also need to be able to prove that the four triangles are equal in area, and so on.

If any of those statements aren't true, then, once again, Pythagoras theorem is in doubt. So they have to be proved as well.

Then, in turn, those statements will rely on other statements which have to be proved and so on ad infinitum.

Obviously we will never get to a starting statement which doesn't have to be proved. So what we do is to take some basic statements as starting statements without proving them. We call these starting statements 'axioms'.

We try to make these axioms as intuitively obvious as possible. Though we have to remember that being intuitively obvious is not the same as being proved. As a result, all we can say about all the mathematical results built on those axioms is that they are true ***if*** the axioms are true.

This method of building mathematical results on a set of axioms by proving each result, then building other results on these results by proof and then further results on those and so on is called 'the axiomatic method' and is the way our body of mathematics is built up.

Alternative axioms

Our system of 2D geometry on a flat plane is based on 5 axioms put together about 2000 years ago by the Greek mathematician, Euclid. They are:

1. A line can be drawn from a point to any other point.
2. A finite line can be extended indefinitely.
3. A circle can be drawn, given a centre and a radius.
4. All right angles are ninety degrees.
5. If a line intersects two other lines such that the sum of the interior angles on one side of the intersecting line is less than the sum of two right angles, then the lines meet on that side and not on the other side.

We can use a different set of axioms and develop a different body of mathematical knowledge, built up in the same way. Of course, it would contain results which we observe not to be true in our experience.

Extending the geometry to three dimensions, using a different version of axiom 5 leads to what are known as non-Euclidean geometries. These might seem purely hypothetical, but some would apply to the universe on a large scale if space-time is curved by gravity rather than being flat. Physicists point at that space-time would be curved unless the density of matter in the universe is finely balanced, though observations do tend to suggest that it is flat or very nearly so.

Incompleteness

It might be nice to think that we can come up with a logical mathematical system based on a given set of axioms in which everything that is true can be proved and in which nothing that is not true can be proved (in other words there are no contradictions).

However, in the 1930s, Kurt Goedel proved that no such system can exist. This is known as the theorem of incompleteness.

At the fundamental level, mathematics can be seen as a branch of philosophy. But most people who use mathematics don't worry about this. Engineers take the approach 'If the maths we use doesn't cause the bridge to fall down, then it's good enough'.