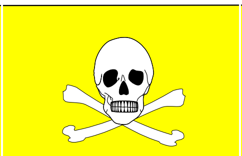
Piracy: A practical Application of Complex Numbers

I learnt about complex numbers at school. 17 years later, when I started teaching maths, I could remember two things. One was that i was the square root of –1. The other was that I spent most of the time wondering why we put so much effort into studying things that didn't exist.

I've now taught complex numbers to another generation and sometimes I wonder whether my students will remember anything more about them 17 years down the track. The problem is that, from a school perspective, not only do complex numbers not exist, but they aren't used for anything either. I told my students that they are used in electronics, fluid mechanics and aerofoil design, but these applications are beyond school maths. I couldn't help feeling that it would be nice to find just one practical problem that can be solved with complex numbers and that is within the reach of high school students. Recently I came across the following.



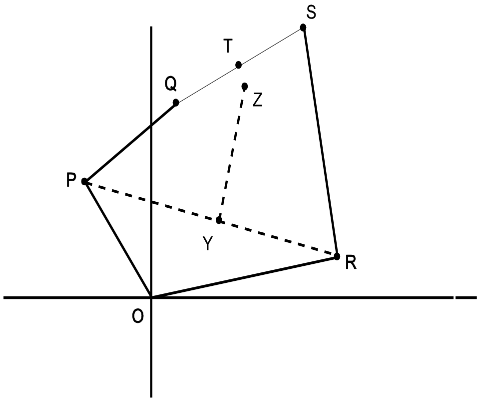
Two pirates landed on a small deserted island to bury some treasure. On the island were just three trees – two large palm trees and a rather dead-looking hibiscusy sort of thing. Otherwise the ground was fairly featurelss grassy sand. They knew that other pirates might come looking for the treasure, so they decided to use a fairly obscure method to position it. Both women (in view of the traditional male domination of the profession, we have to encourage more females to take up piracy) started at the hibiscus bush. The first one paced to one of the palm trees, then turned 90° right and paced the same distance again. The second one paced to the other palm tree, turned 90° left, then paced the same distance again. They then buried the treasure half way between the points they ended up at.

Nine years later, the two pirates returned to the island to dig up the treasure. Because of the action of tides and storms, the island had changed shape a bit. The two palm trees were still there, but there was no trace of the hibiscus.
'How on earth will we find it now?' exclaimed Alice. 'I'm not digging up the whole _@~#★ island!'
'We won't have to,' replied Slasher. 'If we start at one palm tree, walk half way to the other one, then turn left and walk the same distance again, the treasure will be there.'
'But that's not where we buried it,' said Alice. 'Remember we started from that hibiscus bush which ain't here no more.'
'Arrrgh!' said Slasher, 'but the new directions get us to the same spot.'
'How do you know?' said Alice.
'I worked it out with complex numbers the night before we raided that Portugese galleon. Look, I'll show you.'

The problem is to convince Alice that the new directions will locate the treasure.

Slasher was right. Here is her proof.

Set up a complex plane with its origin at the hibiscus bush. P and R are the palm trees, Q and S are the points the pirates paced to. T is the midpoint of QS, the point where they buried the treasure. Y is the mid-point of PR, Z the point where Slasher claims the treasure can be found.



Let the points P and R be represented by the complex numbers u and v respectively.

Then Q = $u+iu$ and S = $v-iv$

T = $\frac{1}{2}(u+iu+v-iv)$

Y = $\frac{1}{2}(u+v)$

Z = $\frac{1}{2}(u+v)-\frac{1}{2}i(v-u)$

= $\frac{1}{2}(u+iu+v-iv)$

= T

So the place she claims the treasure can be found is the place where it is buried.

Here was my practical problem solved with complex numbers. But then I started to wonder why anyone would use complex numbers to solve a problem that could be solved by 'real' methods. So I tried to solve the problem by 'real' methods.

I started with a deductive geometry approach, but had no success, possibly because I didn't know where the hibiscus had been.

Then I tried a coordinate geometry approach. This was more productive, but involved a lot of complicated algebra to get the equations of all the lines involved. I won't take up space with it here, but you can give it a try if you like.

The most efficient 'real' method I found was a vector approach. The vector proof is given below. The diagram is as for the complex number approach except that the plane is the cartesian plane.

Let OP be $a\boldsymbol{i}+b\boldsymbol{j}$ and OR be $c\boldsymbol{i}+d\boldsymbol{j}$

Then PQ = $b\boldsymbol{i}-a\boldsymbol{j}$ and RS = $c\boldsymbol{j}-d\boldsymbol{i}$

OQ = OP +PQ = $a\boldsymbol{i}+b\boldsymbol{j}+b\boldsymbol{i}-a\boldsymbol{j}$

OS = OR +RS = $c\boldsymbol{i}+d\boldsymbol{j}+c\boldsymbol{j}-d\boldsymbol{i}$

OT = ½(OQ +OS) = $\frac{1}{2}(a\boldsymbol{i}+b\boldsymbol{j}+b\boldsymbol{i}-a\boldsymbol{j}+c\boldsymbol{i}+d\boldsymbol{j}+c\boldsymbol{j}-d\boldsymbol{i})$

$= \frac{1}{2}[(a+b+c-d)\boldsymbol{i}+(-a+b+c+d)\boldsymbol{j}$

OY = ½(OP +OR) = $\frac{1}{2}(a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{i}+d\boldsymbol{j})$

PR = OR – OP = $c\boldsymbol{i}+d\boldsymbol{j}-a\boldsymbol{i}-b\boldsymbol{j}$

PY = ½PR = $\frac{1}{2}(c\boldsymbol{i}+d\boldsymbol{j}-a\boldsymbol{i}-b\boldsymbol{j})$

$= \frac{1}{2}[(c-a)\boldsymbol{i}+(d-b)\boldsymbol{j}]$

YZ = $\frac{1}{2}[(c-a)\boldsymbol{j}+(b-d)\boldsymbol{i}]$

$= \frac{1}{2}(c\boldsymbol{j}-a\boldsymbol{j}+b\boldsymbol{i}-d\boldsymbol{i})$

OZ = OY +YZ = $\frac{1}{2}(a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{i}+d\boldsymbol{j}+c\boldsymbol{j}-a\boldsymbol{j}+b\boldsymbol{i}-d\boldsymbol{i})$

$= \frac{1}{2}[(a+b+c-d)\boldsymbol{i}+(-a+b+c+d)\boldsymbol{j}]$

= OT

So the place she claims the treasure can be found is the place where it is buried.

Now I could see why Slasher used complex numbers – the proof was about a third the length of the shortest other method. The main advantage of the complex number method lies in the ease with which rotations can be performed – all we do is multiply by another complex number, in this case either i or $-i$.