Numerical Solution of Differential Equations Using Spreadsheets

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Spreadsheets provide approximate methods for solving differential equations numerically. These methods are analogous to the numerical methods of finding definite integrals, but they can be applied to a much broader range of problems including those which lead to second- and higher-order equations and those which lead to equations in several variables which have to be solved simultaneously.

The solution of such problems is not required by most senior maths syllabuses, yet the mathematics required is less sophisticated than that required for many traditional senior calculus problems which require analytical solutions. Examining such problems in the senior mathematics class has a number of benefits:

- It vastly expands the range of problems which students can model;
- It brings many more meaningful problems within the reach of students;
- It gives experience in meaningful use of technology;
- It introduces students to a technique which is of great importance in applied mathematics;
- It enables students to simulate and experiment with mathematical models of complex situations and to examine the effects of changes in parameters and initial conditions without the need to be able to solve the problems analytically or of laborious re-calculations.
- Use of the spreadsheet facilitates easy graphical presentation of the results, both during experimentation and at completion of the problem.

The remainder of this article consists of notes on the application of spreadsheet techniques to five problems. Problem 1 is a fairly trivial problem which can be solved very simply by analytical means. It is included to demonstrate the technique. The other four problems are of the type to which one might realistically apply such techniques.

The accompanying spreadsheet contains working sheets for each problem. Parameters for each can be adjusted to see the different possible results.

Falling body

A body is dropped. Investigate the distance fallen as a function of time.

Basic equations: $\frac{dv}{dt} = a$ $\frac{ds}{dt} = v$

P1	P2	
t	***v***	***s***
0	0	0
F1	F2	F3
F1	F2	F3
F1	F2	F3
...	...	...

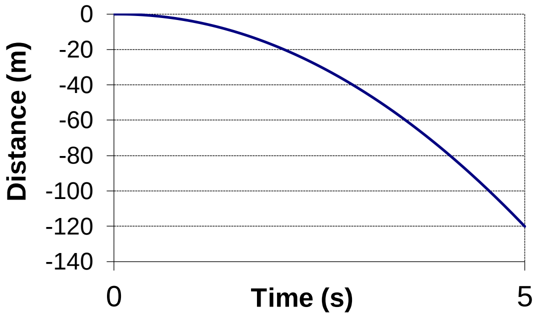
← parameters

← initial conditions

P1: time increment for calculations, say 0.1
P2: acceleration, say –9.8
F1: $t_n = t_{n-1} + 0.1$
F2: $v_n = v_{n-1} + \text{P1*P2}$
F3: $s_n = s_{n-1} + \text{P1*}\ v_{n-1}$

The parameters, P1 and P2, are quantities which the user may wish to vary. They are given their own cells at the top of spreadsheet to make changing them easy, and are referred to by the formulas in the body of the spreadsheet. The formulas, F1 to F3, are spreadsheet formulas. They are not given here in terms of cell references, but in terms of the variables in the situation. They need to be translated into cell-reference formulas when entering the spreadsheet onto a computer.

A graph of s vs t is shown below.



It is easy to calculate that the distance travelled in the first 5 seconds is 122.5. The spreadsheet, however, gives $s = 120.05$ when $t = 5$. This error can be seen in the graph. This is because the velocity during each time increment is taken to be the velocity at the beginning of the time interval. A better result can be obtained by taking the average of the velocities at the beginning and end of the time increment. This can be done by modifying formula F3 to $s_n = s_{n\text{-}1} + \text{P1}^* (v_{n\text{-}1} + v_n) /2$.

The accompanying spreadsheet (Sheet: Falling Body) gives two solutions, the first using the rough method and the second the more refined method (which gives a distance of 122.5 m at 5 seconds). The other for sheets use the rough method, but with more time increments to reduce the error.

Falling body with air resistance

Investigate the motion of a body accelerating under gravity, but with air resistance proportional to the velocity.

Basic equations: $\frac{dv}{dt} = -g - kv \qquad \frac{dh}{dt} = v$

P1	P2	P3	
t	***a***	***v***	***h***
0	F2	P4	P5
F1	F2	F3	F4
F1	F2	F3	F4
F1	F2	F3	F4
...	...	...	...

P1: time increment for calculations
P2: gravitational acceleration
P3: coefficient of air resistance, k
P4: initial velocity
P5: initial height
F1: $t_n = t_{n\text{-}1} + \text{P1}$
F2: $a_n = \text{P2} - \text{P3} * v_n$
F3: $v_n = v_{n\text{-}1} + \text{P1} * a_{n\text{-}1}$
F4: $h_n = h_{n\text{-}1} + \text{P1} * v_n$

Note that the formula for v_n uses a_{n-1} and the formula for h_n uses v_n. I am not sure why, but this seems to produce more far accurate results than using average acceleration and average velocity.

From this spreadsheet, height, velocity and acceleration can be graphed against time or against each other.

Projectile with air resistance

Investigate the motion of a projectile with air resistance proportional to velocity.

Basic equations: $\frac{d^2x}{dt^2} = -kv$ $\qquad \frac{d^2y}{dt^2} = -g - kv$

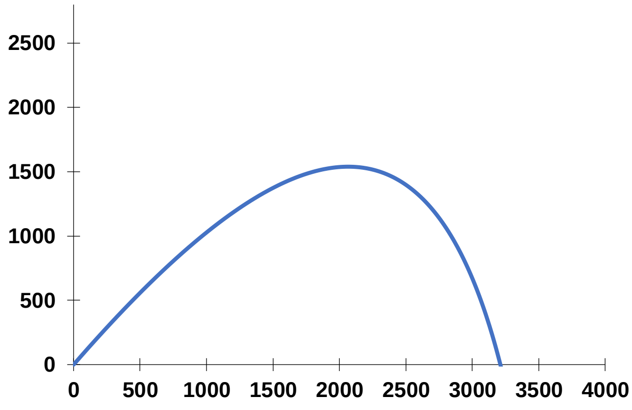
P1	P2	P3	P4	P5		
t	***x"***	***y"***	***x'***	***y'***	***x***	***y***
0	F2	F3	F4a	F5a	P6	P7
F1	F2	F3	F4	F5	F6	F7
F1	F2	F3	F4	F5	F6	F7
F1	F2	F3	F4	F5	F6	F7
…	…	…	…	…	…	…

P1: time increment used for calculations
P2: gravitational acceleration
P3: coefficient of air resistance, k
P4: initial speed
P5: initial direction (degrees elevation)
P6: initial horizontal position
P7: initial vertical position
F1: $t_n = t_{n-1}$ + P1
F2: $x''_n = -$P3 * x'_n
F3: $y''_n =$ P2 – P3 * y'_n
F4a: $x' =$ P4 * cos (P5 * π / 180)
F4: $x'_n = x'_{n-1}$ + P1 * x''_{n-1}
F5a: $y' =$ P4 * sin (P5 * π / 180)
F5: $y'_n = y'_{n-1}$ + P1 * y''_{n-1}
F6: $x_n = x_{n-1}$ + P1 * x'_n
F7: $y_n = y_{n-1}$ + P1 * y'_n

A graph of y vs x gives the trajectory of the projectile. The graph below is for an *initial velocity* of 300 ms^{-1} at 50° elevation, *gravity* = 9.8 ms^{-1} and *coefficient of air resistance* = 0.05.

Students can experiment with varying the conditions and investigate such questions as ‘Is the maximum range still attained by firing the projectile at 45°?’

Other pairs of quantities can be graphed also.



Pendulum

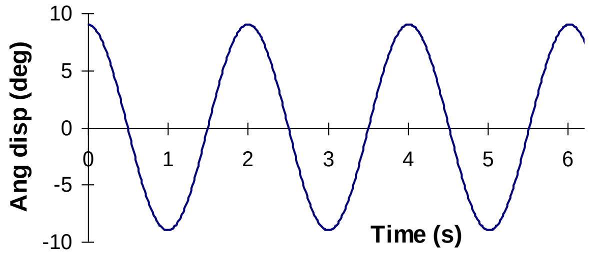
Investigate the motion of a pendulum released from various initial displacements.

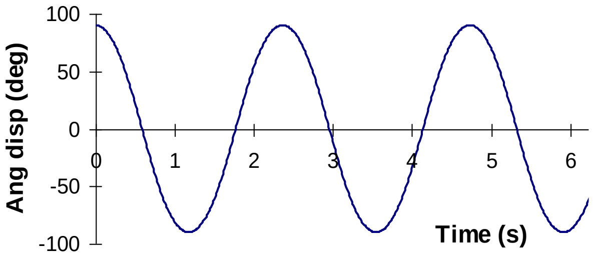
Basic equation: $$\frac{d^2a}{dt^2} = -\frac{g}{l}\sin a$$

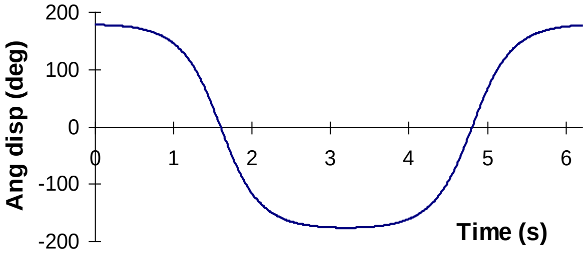
P1	P2	P3	
t	***a”***	***a’***	***a***
0	F2	P4	P5
F1	F2	F3	F4
F1	F2	F3	F4
F1	F2	F3	F4
...	...	...	...

P1: time increment used for calculations
P2: gravitational acceleration
P3: pendulum length
P4: initial angular velocity
P5: initial angular displacement (degrees)
F1: $t_n = t_{n-1} + P1$
F2: $a''_n = P2 / P3 * \sin (a_n * \pi / 180)$
F3: $a'_n = a'_{n-1} + P1 * a''_{n-1}$
F4: $a_n = a_{n-1} + P1 * a'_n$

The following graphs are of angular displacement against time for *time increment* = 0.1, *gravity* = 9.8, *length* = 1m, *initial angular velocity* = 0 and *initial displacement* = 9°, 90° and 177°.







Comet Orbit

Find the trajectory of a comet passing the sun given its initial position and velocity vectors.

Let the sun be at the origin of the plane of the comet's orbit and let the comet be at position (x, y) with velocity (x', y'). As units, we will use Gm (millions of km), kg and days. The mass of the sun is then 2×10^{30} kg and $G = 5 \times 10^{-28}$ Gm^3 kg^{-1} day^{-2}.

Basic equations:

$$\frac{d^2x}{dt^2} = -\frac{1000x}{(x^2+y^2)^{3/2}}$$

$$\frac{d^2y}{dt^2} = -\frac{1000y}{(x^2+y^2)^{3/2}}$$

P1	P2					
t	***x''***	***y''***	***x'***	***y'***	***x***	***y***
0	F2	F3	P3	P4	P5	P6
F1	F2	F3	F4	F5	F6	F7
F1	F2	F3	F4	F5	F6	F7
F1	F2	F3	F4	F5	F6	F7
...	...	...	...	...	...	...

P1: time increment for calculations
P2: Gravitational attraction of orbited body
P3: initial x-component of velocity
P4: initial y-component of velocity
P5: initial x-component of position
P6: initial y-component of position
F1: $t_n = t_{n-1} + \text{P1}$
F2: $x''_n = \text{P2} * x_n / (x^2 + y^2)^{3/2}$
F3: $y''_n = \text{P2} * y_n / (x^2 + y^2)^{3/2}$
F4: $x'_n = x'_{n-1} + \text{P1} * x''_{n-1}$
F5: $y'_n = y'_{n-1} + \text{P1} * y''_{n-1}$
F6: $x_n = x_{n-1} + \text{P1} * x'_n$
F7: $y_n = y_{n-1} + \text{P1} * y'_n$

All orbits produced should be conics. By adjusting the initial conditions, circles, ellipses, parabolas and hyperbolas can be produced. An ellipse and a hyperbola are shown below.