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Algebra as the Study of Relations

Apart from the last couple of sentences, this article was written around the year 2000 while David Ilsley was working for the Queensland School Curriculum Council.

If you want to cheer yourself up, go out onto the street and ask the first person you see about their experience of school algebra. Then watch their eyes light up as they explain to you exactly what algebra is, what it is for, how much they enjoyed learning it and how much difference knowing it has made to their life.

To many students, algebra seems to be a collection of skills without any obvious connection between them except that they all involve letters. As teachers, we can explain that algebra is 'generalised arithmetic', but such an explanation is too abstract to be of help to students.

This article suggests an approach to the teaching of algebra in which algebra is seen as the art of using relations. Students can easily grasp the idea of a relation and the usefulness of relations is more obvious than is the usefulness of manipulating letters.

All the concepts and skills involved in algebra are introduced as means of getting information from relations. Thus students can see the purpose of every new idea and skill as it is introduced.

Background

Traditionally, algebra has been taught by introducing the idea of a pronumeral, like x, then operations on pronumerals to produce expressions like $3x + 2$. This allowed us to ask 'what number is 2 more than $3x$. We then led on to collecting like terms, expanding, factorising, performing inverse operations etc. To most students, this algebra bore no relation to anything else they met in life. They took it on faith that the knowledge was useful – and for most students, the faith was misplaced because they would never use it except in an algebra test.

It is not surprising that when you ask many adults what algebra is about, they cannot give a meaningful answer.

More recently, a different approach has become popular – one in which students observe patterns, then use algebraic expressions to describe those patterns. This approach makes algebra slightly more meaningful for beginners, but it does begin with some fairly difficult concepts (like writing a formula to describe the pattern 4, 7, 10, 13, 16, . . .), and doesn't lead on naturally to other parts of algebra – after this different introduction, many algebra courses then settle back into the old routine of learning about pronumerals, expressions etc. Also, although the sequences of match-stick patterns typically used are more 'concrete' than the pronumerals and expressions of the old approach, they still don't relate very clearly to anything else that might happen in the student's life.

In this article I am going to suggest a different approach. This approach begins with and is based on everyday phenomena; it develops relatively easy concepts before going on to more difficult ones; it develops the concepts which are most likely to be of use in everyday life first; it can be developed progressively from the early grades of primary school; and it leads very naturally into the algebra of senior mathematics.

Most importantly, the approach treats algebra as a convenient means of solving real-life problems. All concepts are developed in practical contexts, and the abstraction which traditionally has made algebra very difficult for many students is avoided.

Some may feel that the essence of algebra is abstraction and that concepts like 'variable' have to have an almost Zen-like undefinability. They may feel that teaching it in a practical life-related manner is missing the point. My view is that if, as a result of studying algebra in Years 1-10, students are well prepared for the algebraic demands of senior mathematics and day-to-day life, then algebra education has served its purpose.

The approach proposed here is based entirely on the concept of a relation, and, in this sense, the study of algebra can be regarded as the study of relations. All the concepts and techniques of algebra are introduced as aids in the use of relations. This way students see the purpose of every idea as it is introduced, rather than having to accept that it will be useful at some later stage.

Introducing relations

Many people would have trouble explaining what algebra is, but the concept of a relation is one that is fairly easy to grasp.

Consider the following information which shows how much Family Payment is received per fortnight by families on $60 000 per year combined income.

A family with no children receives $0
A family with 1 child receives $22.70
A family with 2 children receives $56.85
A family with 3 children receives $161.95
A family with 4 children receives $274.55
A family with 5 children receives $387.15
A family with 6 children receives $499.75

This is the relation between the number of children in a family and the Family Payment received. It is a relation.

Here is another relation – the relation between time and outdoor temperature:

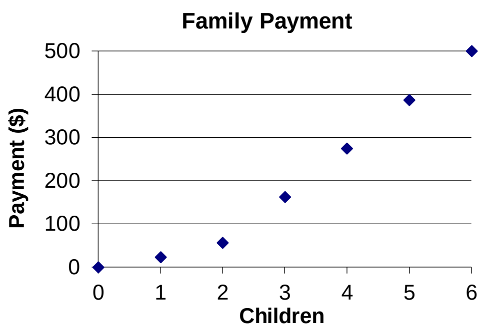
Time	Temperature
5 a.m.	17°
6 a.m.	17°
7 a.m.	18°
8 a.m.	20°
9 a.m.	23°
10 a.m.	25°
11 a.m.	26°
12 noon	27°

A relation is information which allows us to find the value of one quantity given the value of another.

The first example allows us to find the Family Payment, given the number of children, or to find the number of children, given the Family Payment; the second example allows us to find the temperature, given the time, or the time(s), given the temperature.

Relations can be presented in a number of ways:

- as a verbal statement (as in the Family Payment example)
- as a table (as in the temperature example)
- as a graph, eg. the following graph for the Family Payment example



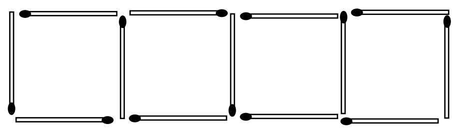
Primary school students have no trouble understanding, dealing with, using and producing such information. They should be able to master the skills of obtaining one quantity given the other in these forms of a relation and of converting from one form of a relation to another, eg. from a table to a graph or vice versa. Practice at this and awareness that they are dealing with relations should provide a good grounding for the more in-depth study of relations in secondary school.

Examples of relations that might be looked at in primary school are:

- Number of points vs placing in a race on sports day
- Number of number facts known vs week
- Height vs age for a plant
- Temperature vs time
- Maximum temperature vs day
- Mass vs height for the children in a class
- Price vs number of sausage rolls
- Hunger vs time
- Boredom vs time
- Time taken vs distance run

Relations with patterns

Consider two relations: one between maximum temperature and day of the month for the first 6 days of a month; and one between the number of adjacent match squares and the number of matches required to make them for 1 to 6 squares.



These relations are given in table form below.

Day	Max temp
1	27
2	24
3	25
4	27
5	28
6	22

No of squares	No of matches
1	4
2	7
3	10
4	13
5	16
6	19

Looking at these relations, students can readily tell that they can predict the number of matches for 7 squares, but that they cannot predict the temperature on the 7th day. This leads to the idea that some relations have a pattern and some do not. Relations with patterns can be expressed in a very concise way – by a formula. In the case of the match-stick example the formula is

***number of matches = number of squares* × 3 + 1**

To begin with, students should be given formulae. Once they are used to the idea, they can begin to develop formulae for proportional and then linear relations themselves.

Introducing students to relations without patterns as well as relations with patters helps them to make a clear distinction between the concepts of ‘relation’ and ‘pattern’.

There is no need at this stage for students to use single-letter abbreviations for variable names. Writing formulae out in English for a while will help to ensure that students understand their meaning.

Discrete and continuous relations

When students present relations as graphs, the issue of discreteness vs. continuity soon arises – students will argue about whether the points should be joined up. This issue can be dealt with at this stage. Students should realise, for example, that a continuous relation can be presented as a graph and perhaps a formula, but not fully as a table. A sample of value pairs needs to be used.

Equations

From their early experiences with relations, students should have no trouble finding one variable given the other, if the relations are expressed as verbal statements, tables or graphs. New skills are needed, however, when the relation is in the form of a formula.

If the formula is explicit with regard to the dependent variable, then finding the value of the dependent variable from the value of the independent variable is the process of substitution. Substitution would be, for most people, the most common use of formulae, and it is appropriate that this skill should be mastered before the other skills of formal algebra.

Substituting for the independent variable in a formula with two variables produces an *explicit* equation in one unknown:

number of matches = number of squares × 3 + 1
⇓
number of matches = 4 × 3 + 1

‘Solving’ this equation is a straightforward application of arithmetic.

On the other hand, substituting for the dependent variable in a formula with two variables produces an *implicit* equation in one unknown:

number of matches = number of squares × 3 + 1
⇓
13 = number of squares × 3 + 1

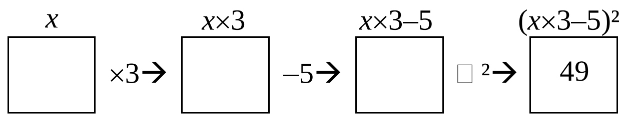
Such implicit equations are what we normally call ‘equations’ and much of algebra consists of learning techniques to solve these.

There are four main ways to solve equations.

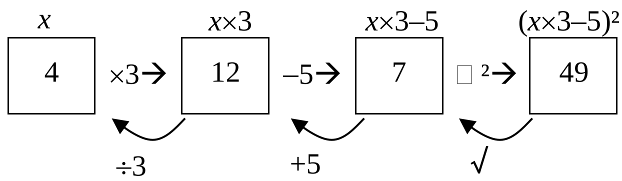
- Guess and check
- Backtracking
- Doing the same thing to both sides (or the balance method)
- Graphing

These are given in order of increasing conceptual difficulty. In using the first three, the fact that the name of the unknown needs to be written several times during the solving of an equations will lead most students to want to abbreviate it. If students decide themselves to abbreviate variable names, the use of symbolism cannot appear mysterious.

Students master ‘guess and check’ quite easily, though they can get frustrated with the laboriousness of it. They are then happy to learn the backtracking method. As the name implies, the backtracking method involves working backwards to the value of the unknown. Students backtrack naturally in their heads to solve equations like $2x + 1 = 5$. For more involved equations, a more formal system needs to be established. For instance, to solve $(3x – 5)^2 = 49$, we draw the following:

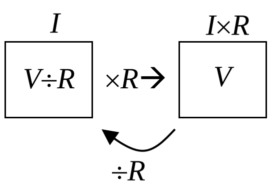


then work backwards filling in the squares as we go.



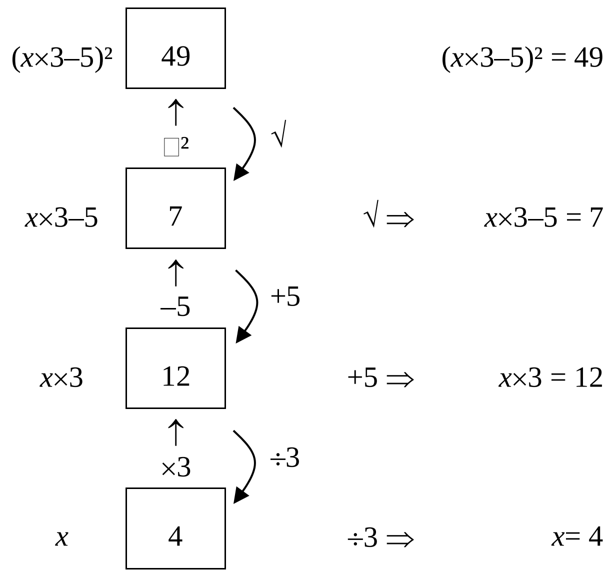
Students can solve quite complex equations easily with this method, provided that they are familiar with inverse operations and the conventions regarding order of operations and provided the unknown occurs only once in the equation.

The method can also be used to change the subject of a formula. For instance, suppose we knew $V = IR$, but we wanted a formula for I. This could be achieved as follows. Treat I, the quantity we want to find, in the same way we treated x in the example above.



Students should use backtracking for quite a while before going on to ‘doing the same thing to both sides’, which is really just a modification of backtracking, but which can handle a wider range of equation types.

If the backtracking is laid out vertically rather than in the traditional horizontal format, it becomes very obvious that doing the same thing to both sides is just another way of laying out the same method.



When the equations become more complex, equation solving provides a meaningful context for manipulation techniques, which, traditionally, have tended to be taught in isolation. These techniques include collecting terms, expanding, operating on rationals etc.

To begin with, equations will be produced by substituting into formulae. A bit later on, students can be introduced to the skill of writing equations directly. This skill is a great asset in senior mathematics and science as well as in several contexts in junior mathematics. Trigonometry is an example. Solving a problem like ‘*If something costs $14.50 after a 30% reduction, how much did it cost before?*’ is another example. The equation would be

$$\textit{initial cost} - \textit{initial cost} \times 0.3 = 14.5$$

or

$$\textit{initial cost} \times 0.7 = 14.5$$

The final and most generally applicable method of solving equations is by graphing. Students graph the expressions represented by each side of the equation and then locate the intersection(s) of the two curves. For many equations that students will come across, this is the only feasible method of solution, for instance we would not expect them to solve a cubic equation with large fractional roots any other way. The graphing method helps reinforce the visual concept of the relations involved.

Graphing should, for the most part be carried out by electronic means. It can be argued that graphing is the most efficacious means of solving quadratic equations and that the time and effort involved in learning analytical methods could be better spent elsewhere, say on functions.

Functions

The concepts of a function as a special and particularly useful type of relation and as a sequence of operations are generally introduced in junior algebra. This can be extended to introduce the most common families of functions, *viz.* constant, proportional, linear, reciprocal, power, exponential, quadratic, polynomial and sinusoidal functions. For each family, students should become aware of some situations in which it arises, its standard form (the idea of a parameter can be useful here), the general shape of the graphs and methods for solving equations derived from the functions. Skills in finding functions to model life-related situations and data can be developed once students have a feel for the main function types.

Explaining why some things always happen

A final application of algebra, which is not generally given a lot of attention in junior mathematics is its use to explain things like the following:

Pick a date on a standard calendar.

S	M	T	W	Th	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

Add together the date above it and the date below it. Then add together the date to the left of it and the date to the right of it. Are the results the same? Will this always happen? Why?

The explanation lies in the relations between the sums and the original date.

Let the original date be d.
Then the first sum a is given by
$$a = d - 7 + d + 7,$$
which simplifies to $a = 2d$.

The second sum b is given by
$$b = d - 1 + d + 1,$$
which simplifies to $b = 2d$.

Leading to senior algebra

Algebra and calculus in senior mathematics are based on the idea of a function. A study of relations like that described above, would provide a good foundation for this.

The algebra required in the less specialised senior courses is restricted to

- substituting into formulae;
- writing and solving equations, eg. to solve trigonometry problems;
- graphing linear functions in linear programming.

These skills are dealt with early on with this approach.

M1 Maths

The sequence of learning in the Algebra strand of M1 Maths is based on the ideas above. Following this sequence through the levels will provide a sound knowledge of the algebra required for high school, the knowledge being based on a common-sense understanding of what it is all about.