

Assignment – Maclaurin Series

Question 1 (*Modelling and Problem Solving*)

Non-polynomial functions can be written as polynomials of infinite degree.

For example, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$

This idea is used to program calculators to calculate cosines (and other functions).

Such polynomials are called power series or Maclaurin series. The power series for any differentiable function of x can be obtained using the formula:

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots, \text{ i.e. } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n,$$

where $f^{(n)}(x)$ is the n th derivative of $f(x)$ and $0!$ is taken to be 1.

Your job is to derive this formula, i.e. to prove that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

To help you, some tips are given in the box below.

1. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$
2. You may assume that the polynomial function which is the best fit to the function $f(x)$
 - has the same value as the function at $x = 0$
 - has the same first derivative as the function at $x = 0$
 - has the same second derivative as the function at $x = 0$
 - and so on
3. Using $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$, find expressions for $f'(x)$, $f''(x)$, $f'''(x)$ and so on.
4. Then find expressions for $f'(0)$, $f''(0)$, $f'''(0)$ etc.
5. Then write $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ in terms of $f'(0)$, $f''(0)$, $f'''(0)$ etc.

Question 2 (*Modelling and Problem Solving*)

(a) Using the result from Question 1, derive the power series for $\cos x$, $\sin x$ and e^x .

(b) Hence show that $e^{i\theta} = \text{cis } \theta$.

This result is known as *Euler's Formula*. It gives a further way of expressing a complex number. $a + bi$ is Cartesian form. $r \text{cis } \theta$ is a polar form known as trigonometric form. $re^{i\theta}$ is another polar form known as exponential form.

(c) Use Euler's Formula to express $4e^{\frac{2\pi}{3}i}$ in trigonometric and Cartesian form.

(d) Use Euler's Formula to express $1 + \sqrt{3}i$ in exponential form.

Question 3 (*Modelling and Problem Solving*)

- (a) Euler's Formula can be expressed more generally as $e^{iz} = \cos z + i \sin z$ where z is any complex number. Use this fact to express $\sin z$ and $\cos z$ in terms of e^{iz} and e^{-iz} .
- (b) *Euler's Identity* relates the five most fundamental numbers of mathematics, 0, 1, i , π and e . It is $e^{i\pi} + 1 = 0$. Pure mathematicians love this equation the way scientists love $E = mc^2$. The following is quoted from Wikipedia.

A reader poll conducted by *Mathematical Intelligencer* named the identity as the most beautiful theorem in mathematics.^[1] Another reader poll conducted by *Physics World* in 2004 named Euler's identity the "greatest equation ever", together with Maxwell's equations.^[2]

The book *Dr. Euler's Fabulous Formula* [2006], by Paul Nahin (Professor Emeritus at the University of New Hampshire), is devoted to Euler's identity; it is 400 pages long. The book states that the identity sets "the gold standard for mathematical beauty."^[3]

Constance Reid claimed that Euler's identity was "the most famous formula in all mathematics."^[4]

Gauss is reported to have commented that if this formula was not immediately apparent to a student on being told it, the student would never be a first-class mathematician.^[5]

After proving the identity in a lecture, Benjamin Peirce, a noted nineteenth century mathematician and Harvard professor, said, "It is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."^[6]

Stanford mathematics professor Keith Devlin says, "Like a Shakespearean sonnet that captures the very essence of love, or a painting that brings out the beauty of the human form that is far more than just skin deep, Euler's equation reaches down into the very depths of existence."^[7]

Use the results from Question 2 to prove Euler's Identity.

Question 4 (*Modelling and Problem Solving*)

Using the results above and showing your reasoning, express the following complex numbers in exact Cartesian form:

- (a) i^i (b) $\ln i$ (c) $\ln(-2)$ (d) i^e
(e) $\cos i$ (f) $\sin(1 + i)$ (g) $\cos^{-1} i$

You may use your calculator to check your results, but not to justify them.

Questions 5 and 6 will be done under test conditions after you hand in your work on Questions 1 to 4. They are to ensure that you have understood the concepts involved in Questions 1 to 4.

Question 5 (*Knowledge and Procedures*)

Prove that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

You may assume the formula for the Maclaurin series expansion of a function.

Question 6 (*Knowledge and Procedures*)

Using the ideas from this assignment and showing your reasoning, express the following complex numbers in exact Cartesian form:

- (a) i^{-i} (b) $\ln(-e)$ (c) $\cos(1-i)$

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Q1.

Let

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$

$$f''(x) = 2a_2 + 2 \times 3a_3x + 3 \times 4a_4x^2 + 4 \times 5a_5x^3 + \dots$$

$$f'''(x) = 2 \times 3a_3 + 2 \times 3 \times 4a_4x + 3 \times 4 \times 5a_5x^2 + \dots$$

$$f''''(x) = 2 \times 3 \times 4a_4 + 2 \times 3 \times 4 \times 5a_5x + \dots$$

$$f'(0) = 1 \times a_1 \quad \therefore a_1 = \frac{f'(0)}{1} = \frac{f'(0)}{1!}$$

$$f''(0) = 1 \times 2a_2 \quad \therefore a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$$

$$f'''(0) = 1 \times 2 \times 3a_3 \quad \therefore a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$$

$$f''''(0) = 1 \times 2 \times 3 \times 4a_4 \quad \therefore a_4 = \frac{f''''(0)}{24} = \frac{f''''(0)}{4!}$$

⋮

etc.

$$\text{Note: } f(0) = a_0 \quad \therefore a_0 = \frac{f(0)}{1} = \frac{f(0)}{0!}$$

$$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^n(0)}{n!}x^n$$

Q2.

a) i)

$$f(x) = \cos x \quad f(0) = 1$$

$$f'(x) = -\sin x \quad f'(0) = 0$$

$$f''(x) = -\cos x \quad f''(0) = -1$$

$$f'''(x) = \sin x \quad f'''(0) = 0$$

$$f''''(x) = \cos x \quad f''''(0) = 1$$

⋮

etc.

$$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\cos x = 1 + 0 + \frac{-1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

ii)

$$f(x) = \sin x \quad f(0) = 0$$

$$f'(x) = \cos x \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad f'''(0) = -1$$

$$f''''(x) = \sin x \quad f''''(0) = 0$$

⋮

etc.

$$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\sin x = 0 + \frac{1}{1!}x + 0 - \frac{1}{3!}x^3 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

iii)

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ f''(x) &= e^x & f''(0) &= 1 \\ f'''(x) &= e^x & f'''(0) &= 1 \\ &\vdots & & \end{aligned}$$

etc.

$$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

b) RTP $e^{i\theta} = \text{cis}\theta$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

sub expansion of $\cos x$ and $\sin x$ into $e^{i\theta}$

$$RHS = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

$$LHS = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \dots$$

$$= 1 + \theta i - \frac{\theta^2}{2!} + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \dots$$

$$= \left(1 + \theta i - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)$$

$$= RHS$$

$$\therefore e^{i\theta} = \text{cis}\theta$$

Q.E.D

c) $4e^{\frac{2\pi}{3}i}$

$$= 4\text{cis}\left(\frac{2\pi}{3}\right)$$

$$= 4\cos\left(\frac{2\pi}{3}\right) + i4\sin\left(\frac{2\pi}{3}\right)$$

$$= 4 \times \frac{-1}{2} + 4 \times \frac{\sqrt{3}}{2}i$$

$$= -2 + 2\sqrt{3}i$$

d) $1 + \sqrt{3}i$

$$= 2\cos\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{3}\right)i$$

$$= 2e^{\frac{\pi}{3}i}$$

Note: need to show conversion between Cartesian and polar

Q3.

$$\text{a) } e^{iz} = \cos z + i \sin z$$

$$e^{-iz} = \cos(-z) + i \sin(-z)$$

$$= \cos z - i \sin z$$

$$e^{iz} + e^{-iz} = \cos z + i \sin z + \cos z - i \sin z$$

$$= 2 \cos z$$

$$\therefore \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$e^{iz} - e^{-iz} = \cos z + i \sin z - \cos z + i \sin z$$

$$= 2i \sin z$$

$$\therefore \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\text{b) RTP } e^{i\pi} + 1 = 0$$

LHS

$$e^{i\pi} + 1$$

$$= \cos \pi + i \sin \pi + 1$$

$$= -1 + 0 + 1$$

$$= 0$$

$$= \text{RHS}$$

\therefore Q.E.D.

Q4.

$$\text{a) } i^i$$

$$= \left(\text{cis } \frac{\pi}{2} \right)^i$$

$$= \left(e^{i\frac{\pi}{2}} \right)^i$$

$$= e^{i^2 \frac{\pi}{2}}$$

$$= e^{-\frac{\pi}{2}}$$

$$\text{b) } \ln i$$

$$= \ln \left(e^{i\frac{\pi}{2}} \right)$$

$$= \frac{\pi i}{2}$$

$$\text{c) } \ln(-2)$$

$$= \ln(2 \text{cis}(\pi))$$

$$= \ln(2e^{i\pi})$$

$$= \ln 2 + i\pi$$

$$\text{d) } i^e$$

$$= \left(\text{cis } \frac{\pi}{2} \right)^e$$

$$= \left(e^{i\frac{\pi}{2}} \right)^e$$

$$= e^{i\left(\frac{\pi}{2}e\right)}$$

$$= \cos\left(\frac{e\pi}{2}\right) + i \sin\left(\frac{e\pi}{2}\right)$$

$$\text{e) } \cos i = \frac{e^{i^2} + e^{-i^2}}{2}$$

$$= \frac{e^{-1} + e^1}{2}$$

$$= \frac{e}{2} + \frac{1}{2e}$$

$$\begin{aligned}
 \text{f) } \sin(1+i) &= \sin 1 \cos i + \cos 1 \sin i \\
 &= \sin 1 \left(\frac{e}{2} + \frac{1}{2e} \right) + \cos 1 \left(\frac{1}{2i} \left(\frac{1}{e} - e \right) \right) \quad * \text{ Note: Need to show } \sin i = \frac{1}{2i} \left(\frac{1}{e} - e \right) \\
 &= \frac{1}{2} \left(e + \frac{1}{e} \right) \sin 1 - \frac{i}{2} \left(\frac{1}{e} - e \right) \cos 1 \quad \text{Note: this simplification is not necessary}
 \end{aligned}$$

$$\begin{aligned}
 * \sin i &= \frac{e^{i^2} - e^{-i^2}}{2i} \\
 &= \frac{e^{-1} - e^1}{2i} \\
 &= \frac{1}{2ei} - \frac{e}{2i} \\
 &= \frac{1}{2i} \left(\frac{1}{e} - e \right) \times \frac{i}{i} \\
 &= -\frac{i}{2} \left(\frac{1}{e} - e \right)
 \end{aligned}$$

$$\text{g) } \cos^{-1} i$$

$$\text{let } z = \cos^{-1} i$$

$$i = \cos z$$

$$i = \frac{e^{iz} + e^{-iz}}{2}$$

$$2i = e^{iz} + e^{-iz}$$

$$2i(e^{iz}) = (e^{iz})^2 + 1 \quad \text{Note: } (e^{-iz})(e^{iz}) = 1$$

$$(e^{iz})^2 - 2i(e^{iz}) + 1 = 0$$

$$e^{iz} = \frac{+2i \pm \sqrt{4i^2 - 4 \times 1 \times 1}}{2}$$

$$= \frac{2i \pm \sqrt{-8}}{2}$$

$$= \frac{2i \pm 2\sqrt{2}i}{2}$$

$$= i \pm \sqrt{2}i$$

$$= (1 \pm \sqrt{2})i$$

$$\therefore iz = \ln[(1 \pm \sqrt{2})i]$$

$$z = \frac{\ln[(1 \pm \sqrt{2})i]}{i} \times \frac{i}{i}$$

$$z = -i \ln[(1 \pm \sqrt{2})i]$$

$$= -i[\ln(1 \pm \sqrt{2}) + \ln i]$$

$$\text{Note: } \ln i = \ln(e^{\frac{\pi}{2}i}) = \frac{\pi}{2}i$$

$$= -\ln(1 \pm \sqrt{2})i - \frac{\pi}{2}i^2$$

$$= \frac{\pi}{2} - \ln(1 \pm \sqrt{2})i$$

$$z = \frac{\pi}{2} - \ln(1 + \sqrt{2})i \quad \text{or } z = \frac{\pi}{2} - \ln(1 - \sqrt{2})i$$

As $\ln(1 - \sqrt{2})$ is imaginary – not in cartesian

$$\frac{\pi}{2} - \ln(1 - \sqrt{2})i$$

$$= \frac{\pi}{2} - [\ln(\sqrt{2} - 1) + \ln(-1)]i$$

$$= \frac{\pi}{2} - [\ln(\sqrt{2} - 1) + i\pi]i$$

$$= \frac{\pi}{2} - i^2\pi - \ln(\sqrt{2} - 1)i$$

$$= \frac{\pi}{2} + \pi - \ln(\sqrt{2} - 1)i$$

$$= \frac{3\pi}{2} - \ln(\sqrt{2} - 1)i$$

Q5.

$$f(x) = \ln(1+x) \quad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \quad f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \quad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \quad f'''(0) = 2$$

$$f^{(4)}(x) = \frac{-6}{(1+x)^4} \quad f^{(4)}(0) = -2 \times 3$$

$$f^{(5)}(x) = \frac{24}{(1+x)^5} \quad f^{(5)}(0) = 2 \times 3 \times 4$$

⋮

etc.

$$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\begin{aligned} \ln(1+x) &= \frac{0}{0!} + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{2 \times 3}{4!}x^4 \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \end{aligned}$$

Q6.

a) i^{-i}

$$= \left(\operatorname{cis} \frac{\pi}{2} \right)^{-i}$$

$$= \left(e^{i\frac{\pi}{2}} \right)^{-i}$$

$$= e^{-i^2\frac{\pi}{2}}$$

$$= e^{\frac{\pi}{2}}$$

b) $\ln(-e)$

$$= \ln e + \ln e^{i\pi}$$

$$= 1 + i\pi$$

c) $\cos i = \frac{e^{ii} + e^{-ii}}{2}$

$$= \frac{e^{-1} + e^1}{2}$$

$$\sin i = \frac{e^{ii} - e^{-ii}}{2i}$$

$$= \frac{e^{-1} - e^1}{2i}$$

$$= \frac{-i}{2}(e^{-1} - e^1)$$

$$\therefore \cos(1-i) = \cos 1 \cos i + \sin 1 \sin i$$

$$= \cos 1 \left(\frac{e^{-1} + e^1}{2} \right) - \frac{i}{2} \sin 1 (e^{-1} - e^1)$$