M1 Maths

N6-3 Matrices

- types of matrix
- matrix operations
- applications of matrices

Summary Learn Solve Revise Answers

This is a very brief introduction to matrices and a few things that can be done with them.

Summary

A **matrix** is a 2-dimensional array of numbers (elements) used for storing related information.

The dimension of a matrix indicates the number of rows and columns, e.g. 4 × 3.

Matrices can be described with various words like row matrix, column matrix, square matrix, identity matrix and zero matrix.

Four operations are defined on matrices: addition, subtraction, multiplication by a scalar and matrix multiplication. Matrix operations can be performed on a graphics calculator.

Matrices can be used to describe various situations and to perform calculations. Two examples are dominance matrices and the solution of sets of simultaneous equations.

Learn

What is a Matrix?

A **matrix** (plural: **matrices**) is a 2-dimensional array of numbers enclosed in brackets used for storing related information.

For instance, we could store the numbers of gold, silver and bronze medals won by Australia, India, the UK and Canada in a matrix like this:

$$\begin{pmatrix} 8 & 6 & 3 \\ 7 & 9 & 5 \\ 8 & 4 & 11 \\ 6 & 11 & 5 \end{pmatrix}$$

Note that, unlike a table, a matrix doesn't have a header row or column explaining what the numbers are. It is just the numbers. This is so that it can be operated on mathematically, e.g. by a graphics calculator or computer. Information about what the numbers are must be included separately from the matrix. In the matrix above, the rows are (from top to bottom) Australia, India, UK, Canada and the columns are (from left to right) gold, silver, bronze.

Matrices have a number of **rows** and a number of **columns**.

3 columns

$$\text{4 rows} \quad \begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} \begin{pmatrix} 8 & 6 & 3 \\ 7 & 9 & 5 \\ 8 & 4 & 11 \\ 6 & 11 & 5 \end{pmatrix}$$

We say the matrix above has 4 rows and 3 columns. We say it is a 4 × 3 matrix. We always give the number of rows first, then the number of columns: remember R C – Roman Catholic. 4 × 3 is the **dimension** (or **order**) of the matrix.

A matrix as a whole can be labelled with a single upper-case letter. So we might call the matrix above '**M**'. In typing, we generally use boldface (not italicised). In handwriting we can use double underline, $\underline{\underline{\mathrm{M}}}$, or a tilde underneath, $\underset{\sim}{\mathrm{M}}$, though this is not essential.

Each entry in a matrix is called an **element** of the matrix. A 4 × 3 matrix has 12 elements. The elements can be labelled, e.g. m_{32} if the element in row 3 and column 2. Remember R C – Roman Catholic. This is 4 in the matrix above. We generally use the lower-case equivalent of the matrix name for the element labels.

A matrix with just one row is called a **row matrix**.

$$\begin{pmatrix} 4 & 0 & -3 & 2.7 & a \end{pmatrix}$$

A matrix with just one column is called a **column matrix** and is the same as a storage vector if you have studied vectors.

$$\begin{pmatrix} 12 \\ 0 \\ 1 \end{pmatrix}$$

A matrix with the same number of rows as columns, e.g. a 3 × 3 matrix, is a sq**uare matrix**.

$$\begin{pmatrix} 5 & 2 & 0 \\ 7 & -3 & 1 \\ -2 & 0 & 6 \end{pmatrix}$$

The **leading diagonal** of a square matrix is the elements along the diagonal from top left to bottom right

The **identity matrix** is a square matrix in which all the elements in the leading diagonal are 1 and all other elements are 0.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A matrix where all the elements are 0 is a **zero matrix**.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Addition and Subtraction of Matrices

Suppose the medal matrix above was for the 2004 Olympics. Suppose we also had the matrix for the 2008 Olympics.

2004 2008

$$\begin{pmatrix} 8 & 6 & 3 \\ 7 & 9 & 5 \\ 8 & 4 & 11 \\ 6 & 11 & 5 \end{pmatrix} \qquad \begin{pmatrix} 3 & 8 & 8 \\ 5 & 6 & 4 \\ 8 & 7 & 10 \\ 5 & 8 & 6 \end{pmatrix}$$

The matrix for the two years combined can be obtained by using the operation of **matrix addition** on the two matrices. To perform this operation, we just add the corresponding elements from the two matrices. So

$$\begin{pmatrix} 8 & 6 & 3 \\ 7 & 9 & 5 \\ 8 & 4 & 11 \\ 6 & 11 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 8 & 8 \\ 5 & 6 & 4 \\ 8 & 7 & 10 \\ 5 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 11 & 14 & 11 \\ 12 & 15 & 9 \\ 16 & 11 & 21 \\ 11 & 19 & 11 \end{pmatrix}$$

If we want to see the improvements from 2004 to 2008, we can use the operation of **matrix subtraction**. We just subtract the elements of the 2004 matrix from the corresponding elements of the 2008 matrix like this

$$\begin{pmatrix} 3 & 8 & 8 \\ 5 & 6 & 4 \\ 8 & 7 & 10 \\ 5 & 8 & 6 \end{pmatrix} - \begin{pmatrix} 8 & 6 & 3 \\ 7 & 9 & 5 \\ 8 & 4 & 11 \\ 6 & 11 & 5 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 5 \\ -2 & -3 & -1 \\ 0 & 3 & -1 \\ -1 & -3 & 1 \end{pmatrix}$$

Not a huge improvement on the whole!

Multiplication by a Scalar

Suppose each country gave their athletes 4 more medals for each medal won in 2004. The matrix for the total number of medals gained could be obtained by using the operation of **multiplication by a scalar**. In this context a scalar just means a number. In this case we will multiply the matrix by 5. We do that simply by multiplying each element by 5.

$$5\begin{pmatrix} 3 & 8 & 8 \\ 5 & 6 & 4 \\ 8 & 7 & 10 \\ 5 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 15 & 40 & 40 \\ 25 & 30 & 20 \\ 40 & 35 & 50 \\ 25 & 40 & 30 \end{pmatrix}$$

Practice

Q1 (a) Put the following data into Matrix M: The town of Drillingham had 5 murders in 2019, 7 in 2020 and 3 in 2021; Mugsville had 12, 5 and 9 in the same years. Then put this data into Matrix A: Drilligham has 12 cases of arson in 2019, 18 in 2020 and 7 in 2021; Mugsville had 14, 12 and 11 in the same years.

(b) Use matrix addition to produce a matrix showing the numbers of murders and arsons combined.

(c) Use matrix subtraction to produce a matrix showing how many more arsons there were than murders.

(d) Each murder caused the population of each town to decrease by 12 people (1 dead, 1 sent to jail and 10 left because the place seemed unsafe). Use multiplication by a scalar to produce a matrix showing the decreases caused by the murders.

Matrix Multiplication

Two matrices can be multiplied together. This operation is a little more complicated than the operations of addition, subtraction and multiplication by a scalar.

Suppose we have matrix T $\begin{pmatrix} 5 & 6 & 3 \\ 7 & 1 & 2 \\ 4 & 8 & 2 \\ 7 & 0 & 4 \end{pmatrix}$ and matrix P $\begin{pmatrix} 3 & 4 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$.

To perform the operation, we take the Row 1 of T and Column 1 of P

$$\begin{pmatrix} 5 & 6 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Then we multiply the first elements in each of these matrices to get 5 × 3 = 15, then the same with the second elements to get 6 × 2 = 12, then the third elements to get 3 × 1 = 3. Then we add these three results to get 15 + 12 + 3 = 30.

Because we used Row 1 and Column 1, this result is put into Row 1 Column 1 of our new matrix.

Then we repeat the above using Row 1 of T and Column 2 of P to get 35, and we put this into Row 1 Column 2 of our new matrix.

Then we repeat with Row 2 of T and Column 1 of P, then Row 2 of T and Column 2 of P, then Row 3 of T and Column 1 of P and so on until we have done every row of T with every column of P. The result will be

$$\begin{pmatrix} 30 & 35 \\ 25 & 32 \\ 30 & 34 \\ 25 & 32 \end{pmatrix}$$

The resulting matrix is called T × P or TP.

Note that the order of the two matrices matters and, in general TP ≠ PT. In other words, matrix multiplication is not commutative.

Also note that we cannot multiply two matrices unless the number of elements in the rows of the first matrix is the same as the number of elements in the columns of the second matrix. Matrices that can be multiplied are called conformable matrices (not to be confused with comfortable mattresses); others are unconformable matrices.

So, $\begin{pmatrix} 5 & 6 & 3 \\ 7 & 1 & 2 \\ 4 & 8 & 2 \\ 7 & 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$ are conformable

$\begin{pmatrix} 3 & 4 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 6 & 3 \\ 7 & 1 & 2 \\ 4 & 8 & 2 \\ 7 & 0 & 4 \end{pmatrix}$ are unconformable

$\begin{pmatrix} 5 & 2 & 0 & 4 \\ 7 & -3 & 1 & 0 \\ -2 & 0 & 6 & 0 \end{pmatrix}\begin{pmatrix} 12 \\ 0 \\ 1 \\ -4 \end{pmatrix}$ are conformable

$\begin{pmatrix} 7 & -3 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 5 & 2 & 0 \\ 7 & -3 & 1 \\ -2 & 0 & 6 \end{pmatrix}$ are unconformable

Practice

Q2 If the following pairs of matrices are conformable, multiply them. If they are unconformable, just say so.

(a) $\begin{pmatrix} 5 & 6 & 3 \\ 7 & 1 & 2 \\ 4 & 8 & 2 \\ 7 & 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 2 & 0 & 4 \\ 7 & -3 & 1 & 0 \\ -2 & 0 & 6 & 0 \end{pmatrix}\begin{pmatrix} 12 \\ 0 \\ 1 \\ -4 \end{pmatrix}$

(c) $\begin{pmatrix} 7 & -3 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 5 & 2 & 0 \\ 7 & -3 & 1 \\ -2 & 0 & 6 \end{pmatrix}$ (d) $\begin{pmatrix} 6 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 12 \\ 0 \\ 1 \\ -4 \end{pmatrix}\begin{pmatrix} 4 & 0 & -3 & 2 \end{pmatrix}$ (f) $\begin{pmatrix} 4 & 0 & -3 & 2 \end{pmatrix}\begin{pmatrix} 12 \\ 0 \\ 1 \\ -4 \end{pmatrix}$

(g) If $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 1 \\ 4 & -1 \end{pmatrix}$, find **AB** and **BA**

Adjacency Matrices and Dominance Matrices

Imagine a group of 6 people, A, B, C, D, E, F. A, B and C are all friends with each other on social media. A is also friends with D and E. D and E are friends and D is friends with F.

This information can be shown on an adjacency matrix like the one below where the rows and columns are in the order A, B, C, D, E, F.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We can see from the matrix that A has the most friends (4) within this group and F has only one.

This matrix is symmetrical about the leading diagonal because social media friendships must be both ways.

A similar type of matrix is a dominance matrix. This might show who beat who in a round-robin tennis competition.

Let’s imagine the same 6 people, A, B, C, D, E and F, all play each other once. We can construct a dominance matrix from the results of the games. It might look like this:

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

It A beats C, this is shown as a 1 in Row 1 (A) Column 3 (C). There must then be a 0 in Row 3 Column 1 because C didn’t beat A.

We can see from the matrix that D won 5 games and therefore wins the prize and that C is just hopeless.

But suppose there is a second prize. Both A and F won 3 games. One way to separate these two is to look at second-order wins, i.e. to see how many people A beat who beat B, how many A beat who beat C etc. This is hard to see by just looking at the matrix, but it can be seen more easily by squaring the matrix, i.e. multiplying it by itself.

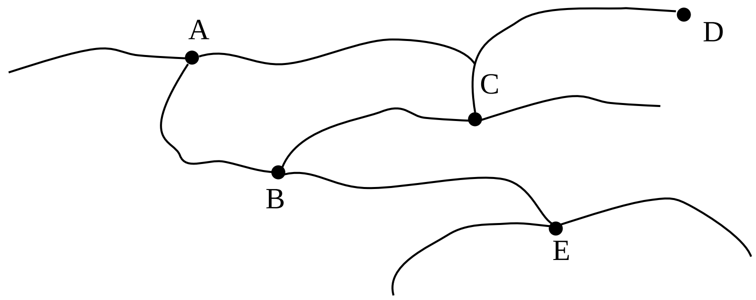
$$\mathbf{D}^2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 1 & 0 \end{pmatrix}$$

$\mathbf{D}^2$ is a second-order dominance matrix and gives the number of second order wins. ($\mathbf{D}$ is a first-order dominance matrix.) Now we can see, for example, that D beat 4 people who beat C. Then we can see that F had 5 second-order wins to A's 4. So the second prize can go to F.

Being quite large matrices, this operation can be done on a graphics calculator. You do have to take the time to enter the matrix, but after that, the calculation is easy.

Practice

Q3 For the following map, produce an adjacency matrix to show which towns are connected directly to which other towns (i.e. without going through a third town) by roads. Use the order A, B, C, D, E.



Q4 Five people play a round-robin chess tournament. A beats B and D, B beats D and E, C beats A, B and E, and D beats A and E, and E beats A. Using first and second-order dominance matrices, rank the 5 players as far as possible.

Solving Simultaneous Equations

Solving 2 simultaneous equations by hand isn't difficult. 3 is more challenging. 4, 5 or more take a long time and few people can do them without making an arithmetic error along the way.

Matrices provide an easier method if you have a graphics calculator.

[Admittedly, graphics calculators have a means of solving multiple simultaneous equations without using matrices, but the matrix method is widely used, especially for very large sets of equations done on computers, so we will look at it.]

Consider the following set of equations:

$2x + 4y - z = 25$

$3x - y + 5z = -4$

$x - 4y + 2z = -20$

The matrix equation

$$\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 25 \\ -4 \\ -20 \end{pmatrix}$$

says exactly the same thing. If you can't see that, just perform the matrix multiplication on the left side of the equations and you will see that it gives the left side of the algebraic equations above.

We then multiply both sides on the left by $\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1}$

to get $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 25 \\ -4 \\ -20 \end{pmatrix}$

then use the calculator to find $\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 25 \\ -4 \\ -20 \end{pmatrix}$.

This gives us $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ So $x = 2$, $y = 5$ and $z = -1$

That was a brief statement of the procedure. A fuller explanation is given in the coloured font below.

> We solve the equation $\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 25 \\ -4 \\ -20 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
>
> But to do this, we first introduce the idea of the inverse of a matrix (strictly speaking the multiplicative inverse).

The inverse of $\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}$, written $\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1}$ is the matrix which when multiplied by it produces the identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The identity matrix is so called because multiplying any matrix by the identity matrix on either side leaves the matrix unchanged (identical).

For example $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 & 0 \\ 7 & -3 & 1 \\ -2 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 0 \\ 7 & -3 & 1 \\ -2 & 0 & 6 \end{pmatrix}$

Inverse matrices are hard to find by hand, but can be found easily on a graphics calculator. Using a calculator

$$\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 18/43 & -4/43 & 19/43 \\ -1/43 & 5/43 & -13/43 \\ -11/43 & 12/43 & -14/43 \end{pmatrix} = \frac{1}{43}\begin{pmatrix} 18 & -4 & 19 \\ -1 & 5 & -13 \\ -11 & 12 & -14 \end{pmatrix}$$

Now we can multiply both sides of our equation

$\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 25 \\ -4 \\ -20 \end{pmatrix}$ on the left by $\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1}$ to get

$$\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 25 \\ -4 \\ -20 \end{pmatrix}$$

This is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 25 \\ -4 \\ -20 \end{pmatrix}$

i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 1 & -4 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 25 \\ -4 \\ -20 \end{pmatrix}$

This is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{43}\begin{pmatrix} 18 & -4 & 19 \\ -1 & 5 & -13 \\ -11 & 12 & -14 \end{pmatrix}\begin{pmatrix} 25 \\ -4 \\ -20 \end{pmatrix}$

Performing the matrix multiplication and then multiplying by $^1/_{43}$, we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

Note that, if two matrices are equal, then all their elements must be equal.

So $x = 2$, $y = 5$ and $z = -1$

Practice

Q5 Solve the following sets of simultaneous equations using matrices.

(a) $2x - 3y + z = -1$

$-x + 2y + 4z = 14$

$5x + y + 3z = -1$

(b) $3a + 4b + c + 2d = 28$

$-2a + b - 4c - 3d = 5$

$2a + 5b - 3c - d = 34$

$a + b + 3c - d = 0$

(c) $-a + 2b - c + 3d = -12$

$3a - 4c + d = 13$

$5a + 2b - c + 2d = 15$

$-2a + 3b + c - 4d = 3$

Solve

Q51 If $\mathbf{A}\begin{pmatrix}-1 & 0\\ 5 & 3\end{pmatrix} = \begin{pmatrix}1 & 3\\ -10 & -6\end{pmatrix}$, find **A**.

Revise

Revision Set 1

Q61 If $\mathbf{A} = \begin{pmatrix}4 & 2 & 1\\ 5 & -3 & -1\\ 2 & 0 & 0\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}-2 & 2 & 0\\ 1 & 0 & 4\\ -2 & 1 & 3\end{pmatrix}$, find by hand

(a) $\mathbf{A} + \mathbf{B}$ (b) $\mathbf{A} - \mathbf{B}$ (c) $-4\mathbf{A}$ (d) $\mathbf{AB}$

Q62 Four people play a round-robin chess tournament. A beats C and D, B beats A and D, C beats B, and D beats C. Using first and second-order dominance matrices, rank the 4 players as far as possible.

Q63 Solve using matrices

$2a + b - 4c + 2d = 17$

$-3a + 2b + 4c - d = -15$

$4b - 3c + 2d = 19$

$2a + 5b - 3c - d = 22$

Answers

Q1 (a) One possible way of arranging the matrices is

$\mathbf{M} = \begin{pmatrix}5 & 12\\ 7 & 5\\ 3 & 9\end{pmatrix}$ $\mathbf{A} = \begin{pmatrix}12 & 14\\ 18 & 12\\ 7 & 11\end{pmatrix}$

(a) $\begin{pmatrix}5 & 12\\ 7 & 5\\ 3 & 9\end{pmatrix} + \begin{pmatrix}12 & 14\\ 18 & 12\\ 7 & 11\end{pmatrix} = \begin{pmatrix}17 & 26\\ 25 & 17\\ 12 & 20\end{pmatrix}$

(b) $\begin{pmatrix}12 & 14\\ 18 & 12\\ 7 & 11\end{pmatrix} - \begin{pmatrix}5 & 12\\ 7 & 5\\ 3 & 9\end{pmatrix} = \begin{pmatrix}7 & 2\\ 11 & 7\\ 4 & 2\end{pmatrix}$

(c) $\begin{pmatrix} 5 & 12 \\ 7 & 5 \\ 3 & 9 \end{pmatrix} \times 12 = \begin{pmatrix} 60 & 144 \\ 84 & 60 \\ 36 & 108 \end{pmatrix}$

Q2 (a) $\begin{pmatrix} 30 & 35 \\ 25 & 32 \\ 30 & 34 \\ 25 & 32 \end{pmatrix}$ (b) $\begin{pmatrix} 46 \\ 85 \\ -18 \end{pmatrix}$ (c) unconformable

(d) $\begin{pmatrix} -11 & 25 \\ -1 & -1 \end{pmatrix}$ (e) $\begin{pmatrix} 48 & 0 & -36 & 24 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & -3 & 2 \\ -16 & 0 & 12 & -8 \end{pmatrix}$ (f) (37)

(g) $\mathbf{AB} = \begin{pmatrix} 7 & 5 \\ -6 & -3 \end{pmatrix}$ $\mathbf{BA} = \begin{pmatrix} 13 & -9 \\ 14 & -9 \end{pmatrix}$

Q3 $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

Q4 $\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\mathbf{D}^2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

By **D**, the ranking is C first with 3 wins, then A, B and D on 2 wins, then E on 1 win.

By $\mathbf{D}^2$, A and D have 4 second-order wins and B has 3.

So the overall ranking is C first, then A and D, then B, then E.

Q5 (a) $x = -2$, $y = 0$, $z = 3$

(b) $a = 2$, $b = 5$, $c = -2$, $d = 1$

(c) $a = 4$, $b = 0$, $c = -1$, $d = -3$

Q51 $A = \begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix}$

Q61 (a) $\begin{pmatrix} 2 & 4 & 1 \\ 6 & -3 & 3 \\ 0 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 0 & 1 \\ 4 & -3 & -5 \\ 4 & -1 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} -16 & -8 & -4 \\ -20 & 12 & 4 \\ -8 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} -8 & 9 & 11 \\ -11 & 9 & -15 \\ -4 & 4 & 0 \end{pmatrix}$

Q62 $\mathbf{D} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $\mathbf{D}^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Therefore the ranking is: B first, A second, C and D equal third.

Q63 $a = 2$, $b = 2$, $c = -3$, $d = 1$