

M1 Maths

N6-2 Vectors

- scalars and geometric vectors in 2 and 3 dimensions
- addition, subtraction, multiplication by a scalar
- scalar and vector products
- storage vectors

[Summary](#) [Learn](#) [Solve](#) [Revise](#) [Answers](#)

This module is an introduction to vectors and may not go as far as is required in some specialised high-school maths courses. It should, however, provide all the vector knowledge that is required for high-school physics courses.

The module is still under construction

Summary

Scalars, vectors, mass, energy, displacement, velocity, acceleration, force, momentum

Vector diagrams

Addition, subtraction, multiplication by a scalar (division)

Resolution

Cartesian form, conversion

Unit vectors

3D vectors

Scalar product

Angles, distance on Earth

Vector product

Storage vectors

Learn

Scalars and Vectors

Speed is a measure of how fast something is moving. It can be specified with a single number, e.g. 5 m/s. The speedometer on a car measures the car's speed.

The word 'velocity' has a different meaning: it is a measure of how fast something is moving and in what direction. It takes more than one number to specify it. For example it might be 5 m/s on a bearing of 110° . An air traffic control radar would measure a plane's velocity.

Velocity has a magnitude (e.g. 5 m/s) and a direction (e.g. 110°). Speed just has a magnitude; the direction is not specified.

A quantity like velocity with a magnitude and a direction is called a vector (or vector quantity). A quantity like speed with just a magnitude is called a scalar (or scalar quantity).

Displacement is another vector; distance is the corresponding scalar. So we can talk of a distance of 3 km or a displacement of 3 km to the south.

Acceleration is a vector. For example, we might say a car is accelerating at 2 m/s/s or 2 m/s^2 in a direction of 125° . We do tend to talk about a car accelerating at 2 m/s/s without specifying the direction. Strictly speaking, we are then just giving the magnitude of the acceleration. In this sense we are using acceleration as a scalar, though in maths, science, engineering etc. acceleration is considered to always be a vector.

In the same way, force is a vector, e.g. 10 N downwards, though sometimes we might just specify the magnitude as 10 N if the direction is understood or doesn't matter.

Momentum is another vector and there are others like electrical potential gradient etc., which we won't consider here.

Vectors have a lot of application in physics. If you are studying physics, you will meet these ideas there too.

Note on direction

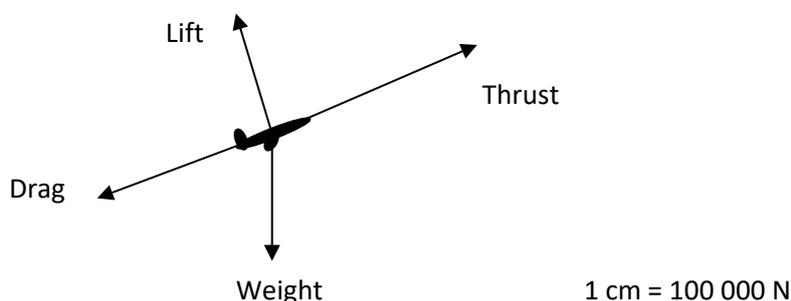
As with other areas of maths, when we say a direction of 110° , this could be used two different ways: in the geographic sense of a bearing (angle clockwise from North), or in the mathematical sense (angle anti-clockwise from the x -axis). In this module, the geographical sense will be used if the context implies that the directions are north, east etc. If there is no such implication, then the mathematical sense will be used.

Vector Diagrams

Vectors can be specified numerically, e.g. 10 m towards 135° . But it is often useful to represent them diagrammatically. We do this with an arrow. The arrow has its back

end at the point where it applies, it points in the same direction as the vector and its length represents the magnitude of the vector.

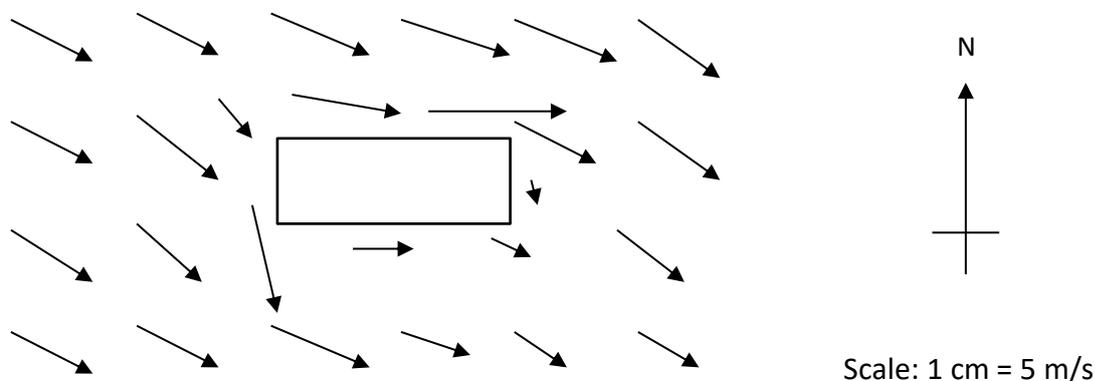
One commonly used type of vector diagram is a force diagram showing the forces acting on an object. The forces on an aeroplane ascending after take-off might look like this:



All forces act on the plane, so all the arrows have their tail at the plane and extend in the direction of the force. The scale can be used to read off the magnitude of each force and a protractor could be used to read off the direction.

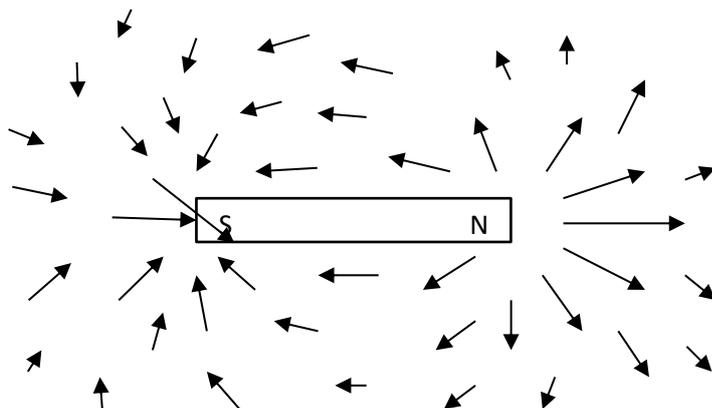
Often, though, such force diagrams are used just to show the number and arrangement of forces acting on a body and so no scale is given. However, the lengths of the arrows should still be proportional to the magnitudes of the vectors, so that large forces are represented as longer arrows, smaller forces as smaller arrows etc.

Another type of vector diagram shows the values at different places of a vector field. A vector field consists of the values of a particular vector quantity at different places. Wind velocity 2 m above the ground at different places around a building would be a vector field. A diagram of the wind velocity at different points around a building, as seen from above, might look like this:



Note that the field has a value at all points around the building, but putting an infinite number of arrows on the diagram would make it useless, so a representative selection of points is used.

The magnetic force field around a bar magnet might look like this:



Practice

- Q1 Sketch a vector diagram showing the forces acting on a cat sitting on flat ground in the wind. Your diagram should show gravity, the upward reaction of the ground which counters gravity to the cat doesn't sink into the ground, the force of the wind and friction.
- Q2 Draw a vector diagram of a brick sitting on sloping ground.
- Q3 In the magnet diagram above, where is the magnetic field strongest?
- Q4 In the wind diagram above, where is the wind blowing most slowly? Why would it be slowest here?
- Q5 In the wind diagram, in which direction is the air moving along each of the four walls?
- Q6 Draw a vector diagram of the gravitational force field around the Earth.
- Q7 Draw a vector diagram of the wind velocity around a tornado.

Vector Notation

The displacement vector 4 m towards 135° can be written as (4 m, 135°). Note that magnitude always comes first, followed by direction, and that including the $^\circ$ symbol helps avoid any confusion with coordinates or other types of ordered pair.

It is often convenient to refer to a vector using a single letter. For example, we might call the displacement 'vector 4 m east' \mathbf{a} and the displacement 'vector 3 m north' \mathbf{b} .

Then $\mathbf{a} = (4 \text{ m}, 090^\circ)$, $\mathbf{b} = (3 \text{ m}, 000^\circ)$.

It is convention for use lower-case letters for vectors. This can distinguish them for matrices which are generally represented by upper-case letters.

Note that we use bold type for vectors to distinguish them from scalars. However, when writing by hand, it is hard to produce bold letters, so we place a squiggle underneath the letter like this:

$$\underset{\sim}{a} \quad \underset{\sim}{b} \quad \underset{\sim}{u} \quad \underset{\sim}{v}$$

Or, less commonly, we place an arrow above the letter, like this:

$$\vec{a} \quad \vec{b} \quad \vec{u} \quad \vec{v}$$

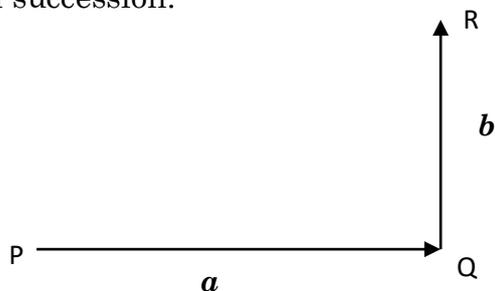
Including the squiggle (or arrow) is quite important, but easy to forget. Try to train yourself.

The magnitude of vector \mathbf{a} is written $|\mathbf{a}|$ with bars either side of the vector.

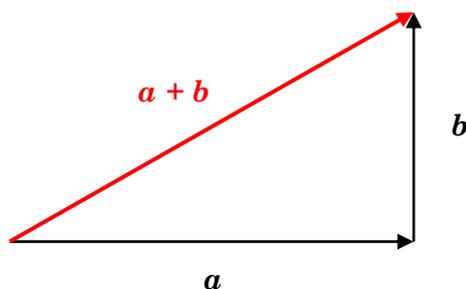
Adding Vectors

Consider the displacement vectors mentioned above: $\mathbf{a} = 4$ m east; $\mathbf{b} = 3$ m north

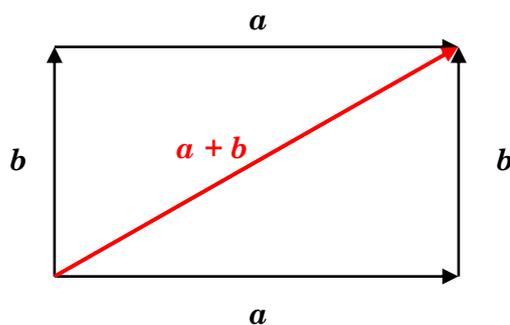
The sum $\mathbf{a} + \mathbf{b}$ is defined as the displacement that would result from performing the two displacements in succession.



It would be the move from P to Q, then from Q to R, i.e. the move from P to R. This would be a displacement of about 5 m in a direction roughly northeast.



As you can see below, it doesn't matter which displacement happens first. $\mathbf{a} + \mathbf{b}$ is always equal to $\mathbf{b} + \mathbf{a}$. In other words vector addition is commutative.



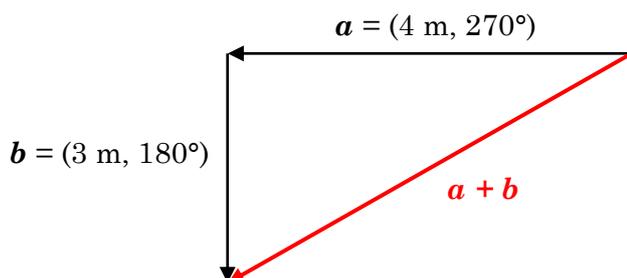
So vectors can be added using a diagram as above. If we need a numerical value for the sum, we can use a bit of Pythagoras and trigonometry. In the case above, this is simple because \mathbf{a} and \mathbf{b} are perpendicular.

$$|\mathbf{a} + \mathbf{b}| = \sqrt{4^2 + 3^2} = 5 \text{ m} \quad \text{and the bearing of } \mathbf{a} + \mathbf{b} \text{ is } \tan^{-1} \frac{4}{3} = 53^\circ.$$

(Remember $|\mathbf{a} + \mathbf{b}|$ means the magnitude of $\mathbf{a} + \mathbf{b}$)

So $\mathbf{a} + \mathbf{b} = (5 \text{ m}, 53^\circ)$.

Vectors will not always be in the first quadrant. Thus it is always a good idea to do a sketch diagram to avoid mistakes, particularly with direction. For instance to add \mathbf{a} (4m, 270°) and \mathbf{b} (3m, 180°), you should draw a diagram something like this:



The magnitude will be 5 m. The top-right angle of the triangle will be 37° , so the direction of the vector will be 233° .

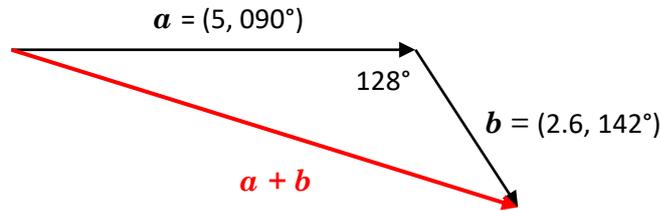
Practice

Q8 Find $\mathbf{a} + \mathbf{b}$ if

- (a) $\mathbf{a} = (5\text{m}, 090^\circ)$, $\mathbf{b} = (2\text{m}, 000^\circ)$
- (b) $\mathbf{a} = (6\text{m}, 090^\circ)$, $\mathbf{b} = (2\text{m}, 180^\circ)$
- (c) $\mathbf{a} = (3\text{m}, 270^\circ)$, $\mathbf{b} = (2\text{m}, 000^\circ)$

(d) $\mathbf{a} = (4\text{m}, 180^\circ)$, $\mathbf{b} = (2\text{m}, 270^\circ)$

If the vectors are not perpendicular, then the sine or cosine rule has to be used. If \mathbf{a} is 5 m towards 090° and \mathbf{b} is 2.6 m towards 142° as in the diagram below,



then the magnitude of $\mathbf{a} + \mathbf{b}$ can be calculated using the cosine rule and the top-left angle in the triangle can then be calculated using the sine rule. From that, the bearing of $\mathbf{a} + \mathbf{b}$ can be found. $\mathbf{a} + \mathbf{b} = (6.91 \text{ m}, 107^\circ)$. (See Module M5-2 (Solving Triangles) if you need a refresher on the sine and cosine rules.)

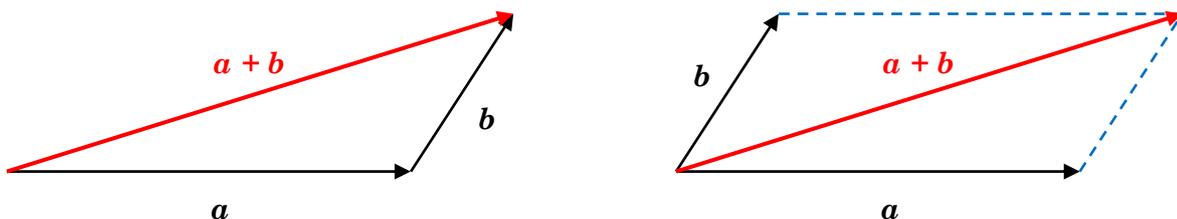
Again, not all vectors will be in the first quadrant, so always draw a sketch diagram

Practice

Q9 Find $\mathbf{a} + \mathbf{b}$ if

- (a) $\mathbf{a} = (5\text{m}, 090^\circ)$, $\mathbf{b} = (4\text{m}, 060^\circ)$
- (b) $\mathbf{a} = (6\text{m}, 090^\circ)$, $\mathbf{b} = (2\text{m}, 120^\circ)$
- (c) $\mathbf{a} = (3\text{m}, 180^\circ)$, $\mathbf{b} = (2\text{m}, 210^\circ)$
- (d) $\mathbf{a} = (4\text{m}, 290^\circ)$, $\mathbf{b} = (7\text{m}, 76^\circ)$

Note that diagrams for addition of vectors can be formed by placing the two vectors being added 'head to toe' and connecting the beginning to the end as in the diagram on the left below, or by making them two adjacent sides of a parallelogram and drawing the diagonal as in the diagram on the right below.



These are sometimes called the triangle method and the parallelogram method.

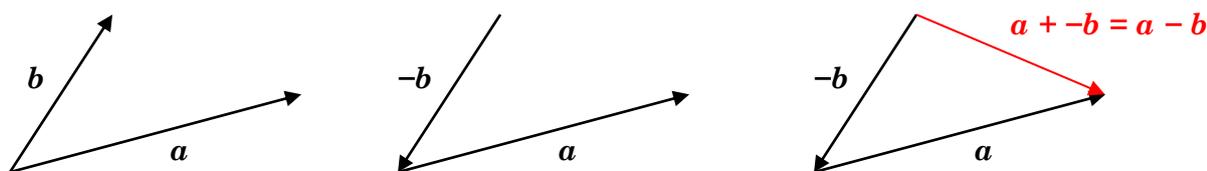
Negative of a vector

$-a$, the negative of vector a is defined as the vector with the same magnitude as a , but pointing in the opposite direction.

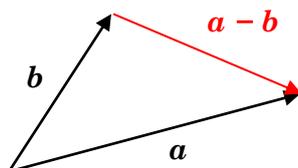


Subtracting vectors

Vector subtraction is defined as addition of the negative. So $a - b = a + (-b)$.



Note that, if a and b start from the same point, then $a - b$ is the vector from the end of b to the end of a . This provides an easier way of drawing $a - b$.



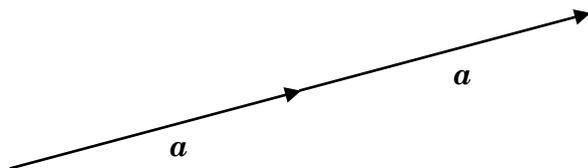
The magnitude and direction of $a - b$ can be calculated from the magnitudes and directions of a and b using trigonometry in a way similar to $a + b$.

Practice

Q10 Find

Multiplication by a scalar

Clearly, $\mathbf{a} + \mathbf{a}$ is a vector with the same direction as \mathbf{a} , but twice the magnitude.



Similarly, $\mathbf{a} + \mathbf{a} + \mathbf{a} + \mathbf{a} + \mathbf{a}$ would be a vector in the same direction, but with 5 times the magnitude.

We can write $\mathbf{a} + \mathbf{a} + \mathbf{a} + \mathbf{a} + \mathbf{a}$ as $\mathbf{a} \times 5$ or $5\mathbf{a}$.

This defines the operation of multiplying a vector by a number (or scalar):

$n\mathbf{a}$ where n is a number (or scalar) and \mathbf{a} is a vector
is the vector with the same direction as \mathbf{a} , but n times the magnitude

We don't normally talk about dividing vectors by numbers. Instead, we just multiply by the reciprocal of the number. So $\mathbf{a} \div 2 = \frac{1}{2}\mathbf{a}$.

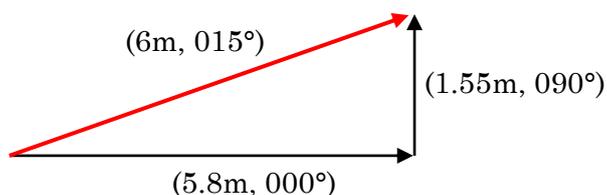
Practice

Q11 Find

Polar and Cartesian Forms

It is often useful to express a vector as the sum of two vectors at right angles, for instance a horizontal vector and a vertical vector or a vector pointing east and a vector pointing north or a vector in the x -direction and a vector in the y -direction.

For example, the vector $(6\text{m}, 015^\circ)$ (using direction here in the mathematical sense of rotation anti-clockwise from the x -axis) is the sum of $(5.80\text{m}, 000^\circ)$ and $(1.55\text{m}, 090^\circ)$.



Expressing a vector as the sum of two vectors at right angles is called expressing it in Cartesian form.

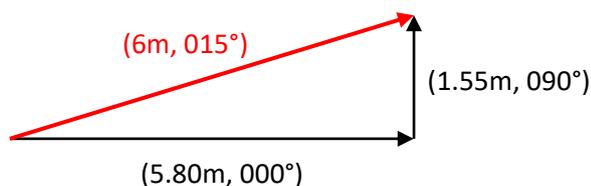
The vector above is written in Cartesian form as $\begin{pmatrix} 5.80 \\ 1.55 \end{pmatrix}$.

Conventionally, the first number is the magnitude of the vector in the x -direction (or horizontal or easterly direction) and the second number is the magnitude of the vector in the y -direction (or vertical or northerly direction). These are called the x -component and y -component of the vector respectively.

To distinguish it, the $(6\text{m}, 015^\circ)$ form of the vector is called polar form. Polar form gives the magnitude and direction; Cartesian form gives the perpendicular components.

Conversion between Polar and Cartesian Forms

Conversion involves Pythagoras and trigonometry on the triangle below.



Starting with the vector $(6\text{m}, 015^\circ)$, the x -component is $6 \cos 15^\circ (= 5.80)$ and the y -component is $6 \sin 15^\circ (= 1.55)$,

So the vector in Cartesian form is $\begin{pmatrix} 5.80 \\ 1.55 \end{pmatrix}$.

Starting with the vector $\begin{pmatrix} 5.80 \\ 1.55 \end{pmatrix}$, by Pythagoras, the magnitude is $\sqrt{5.80^2 + 1.55^2} = 6.00$, and the direction is $\tan^{-1}(1.55 \div 5.80)$, which is 015° .

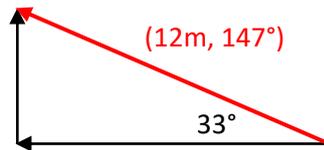
So the vector in polar form is $(6\text{m}, 015^\circ)$.

Practice

Q12 Find

If the vector is not in the first quadrant, it is safest to draw the diagram, work out lengths and angles in triangles, then work out the required numbers from that.

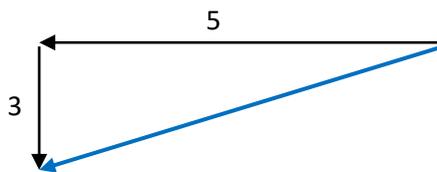
For example, to convert $(12, 147^\circ)$ to Cartesian form, we sketch the diagram:



The length of the base of the triangle is $12 \cos 33^\circ = 10.06$
and the length of the height of the triangle is $12 \sin 33^\circ = 6.54$.

Looking at the diagram, we see that the x -component is negative and the y -component is positive, so the Cartesian form is $\begin{pmatrix} -10.06 \\ 6.54 \end{pmatrix}$.

To convert $\begin{pmatrix} -5 \\ -3 \end{pmatrix}$ to polar form, again, draw the triangle.



The magnitude is then $\sqrt{5^2 + 3^2} = 5.83$

The top-right angle in the triangle is $\tan^{-1} 3/5 = 31^\circ$

So the vector is $(5.83, 211^\circ)$.

Practice

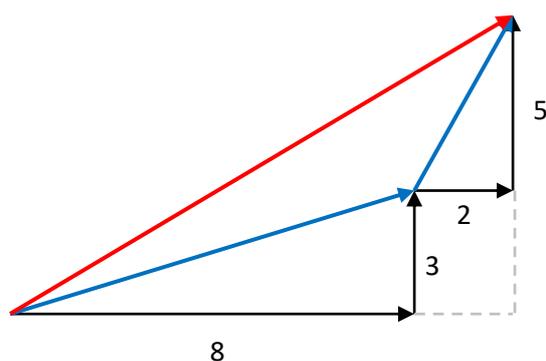
Q13 Find

Using Cartesian Form

The Cartesian form of vectors is the more useful form in many situations. The Cartesian form makes adding and subtracting vectors easy. In fact, rather than use the cosine and sine rules to add vectors, it can be easier to convert them to Cartesian form, then add, then convert back.

For example, to add the vectors $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$, we just add the x -components and add the y -components to get $\begin{pmatrix} 10 \\ 8 \end{pmatrix}$.

The diagram below should make it obvious why this works.



The same applies if some of the components are negative, so $\begin{pmatrix} 4 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$

For subtraction, just subtract the components, so $\begin{pmatrix} 15 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ -4 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}$

Multiplication by a scalar is similarly easy: $\begin{pmatrix} 4 \\ -1 \end{pmatrix} \times 2.5 = \begin{pmatrix} 10 \\ -2.5 \end{pmatrix}$

Practice

Q14 Find

Unit vectors

Sometimes it is handy to use unit vectors. A unit vector is a vector of magnitude 1.

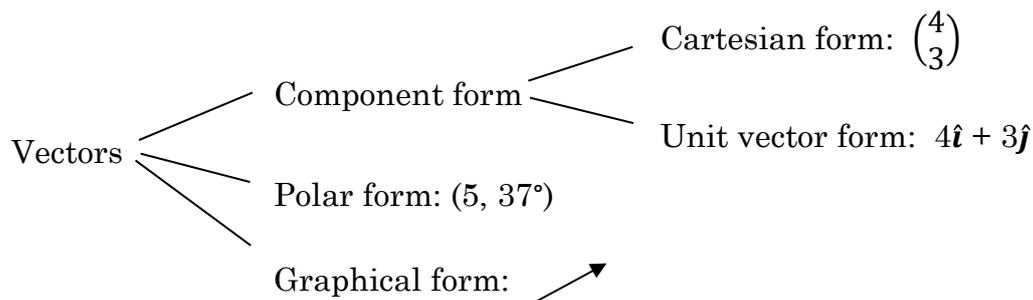
\hat{i} is the unit vector pointing in the positive x -direction

\hat{j} is the unit vector pointing in the positive y -direction.

In handwriting, these are written with a squiggle underneath and a cap on top.

Using these, we can write the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ as $4\hat{i} + 3\hat{j}$ and the vector $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$ as $-\hat{i} - 5\hat{j}$.

We call this unit vector form. So there are three written forms for a vector – polar, Cartesian and unit vector, as well as the graphical form using an arrow. ‘Component form’ refers to both Cartesian form and unit vector form.



Practice

Q15 Express the following vectors in unit vector form:

- | | | | |
|--|---|--|---|
| (a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ | (b) $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ | (c) $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ | (d) $\begin{pmatrix} -3.5 \\ 0 \end{pmatrix}$ |
| (e) $(12, 045^\circ)$ | (f) $(4, 270^\circ)$ | (g) $(1, 180^\circ)$ | (h) $(5.2, 204^\circ)$ |

Q16 Express the following vectors in Cartesian form::

- | | | | |
|--------------------------|----------------------------|-----------------|-------------------------------|
| (a) $3\hat{i} - \hat{j}$ | (b) $-2\hat{i} + 5\hat{j}$ | (c) $-3\hat{j}$ | (d) $4.2\hat{i} + 2.7\hat{j}$ |
|--------------------------|----------------------------|-----------------|-------------------------------|

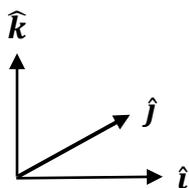
Q17 Express the following vectors in polar form

- | | | | |
|---------------|-------------------------|---------------------------|-----------------------------|
| (a) \hat{i} | (b) $\hat{i} - \hat{j}$ | (c) $3\hat{i} + 2\hat{j}$ | (d) $-\hat{i} + 0.5\hat{j}$ |
|---------------|-------------------------|---------------------------|-----------------------------|

Three-Dimensional Vectors

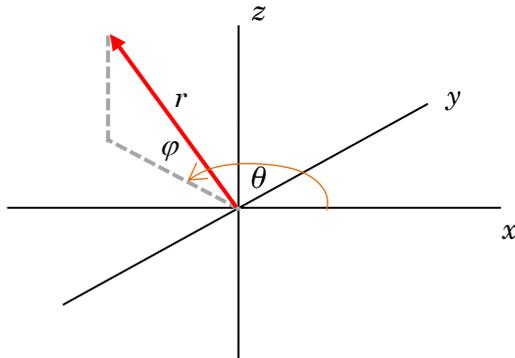
So far, we have considered vectors in a plane. We can consider vectors in three dimensions by adding a third unit vector perpendicular to \hat{i} and \hat{j} . As you might expect, we call it \hat{k} .

If \hat{i} is the unit vector in the x -direction on a horizontal plane, and \hat{j} is the unit vector in the y -direction on the same horizontal plane, then \hat{k} is the unit vector vertically upwards (in the z -direction).



Using this idea, we can describe a displacement to a point 200 m further east, 120 m further south and 80 m higher as $220\hat{i} - 120\hat{j} + 80\hat{k}$ in unit vector form, or as $\begin{pmatrix} 200 \\ -120 \\ 80 \end{pmatrix}$ in Cartesian form.

Polar form for 3D vectors is most commonly given as a magnitude, r , and two angles, θ and φ (Greek letters theta and phi). θ is the angle anti-clockwise from the x -axis in the x - y plane or east in the horizontal plane; φ is the angle upwards from the x - y plane or the horizontal plane.



The vector shown might be $(10 \text{ m}, 140^\circ, 28^\circ)$ in polar form.

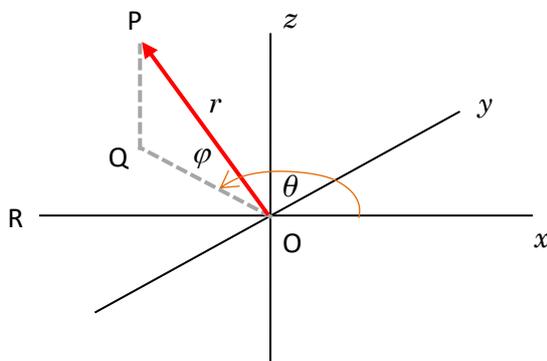
Converting 3D Vectors

Converting between polar and Cartesian form for 3D vectors is a similar process to conversions in 2D, though of course with extra steps. It basically involves right-angle trigonometry and Pythagoras.

There are formulae for the conversions, but they are not easy to remember and, for most people it is best to treat the conversions as exercises in trig and Pythagoras.

Polar to Cartesian Form

Taking the vector above, $(10 \text{ m}, 140^\circ, 28^\circ)$, we can proceed like this.



The z -component is $10 \sin 28^\circ = 4.69$

$OQ = 10 \cos 28^\circ = 8.83$

The y -component is $8.82 \sin 40^\circ = 5.67$

The x -component is $8.82 \cos 40^\circ = 6.76$, but negative

So the vector is $\begin{pmatrix} -6.76 \\ 5.67 \\ 4.69 \end{pmatrix}$ or $-6.76\hat{i} + 5.67\hat{j} + 4.69\hat{k}$

Cartesian to Polar Form

Going the other way, starting with $-6.76\hat{i} + 5.67\hat{j} + 4.69\hat{k}$,

The magnitude is $6.76^2 + 5.67^2 + 4.69^2 = 10$

$$\varphi = \sin^{-1}(4.69 / 10) = 28^\circ$$

$$OQ = 10 \cos 28^\circ = 8.83$$

$$\sin ROQ = 5.67 / 8.83 = 0.642 \Rightarrow ROQ = 40^\circ \Rightarrow \theta = 140^\circ$$

So the vector is (10 m, 140° , 28°).

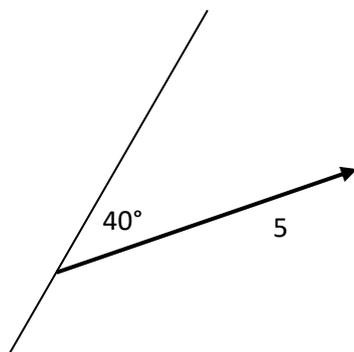
Make sure you can follow the steps above, then try the following until you are confident with the procedures. If you are reasonable at solving multi-step trig problems, you should be able to work out how to do them even if you forget.

Practice

Q18 Find

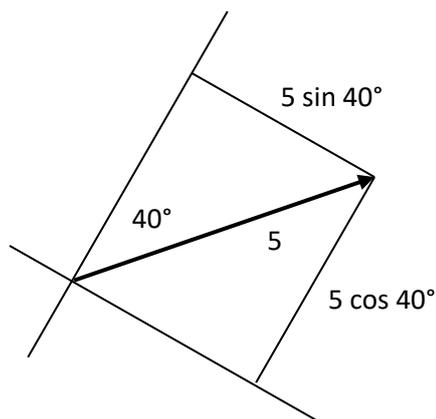
Component of a vector in any direction

We have looked at the components of a vector in the x and y ($\pm z$) directions. Sometimes it is useful to find the component of a vector in some more general direction. For instance the component of the vector $(5, 20^\circ)$ in the direction 60° anticlockwise from the x -axis.



The angle between the vector and the direction we want the component in is 40° .

Consider writing the vector as components in the 60° direction and the direction perpendicular to that.

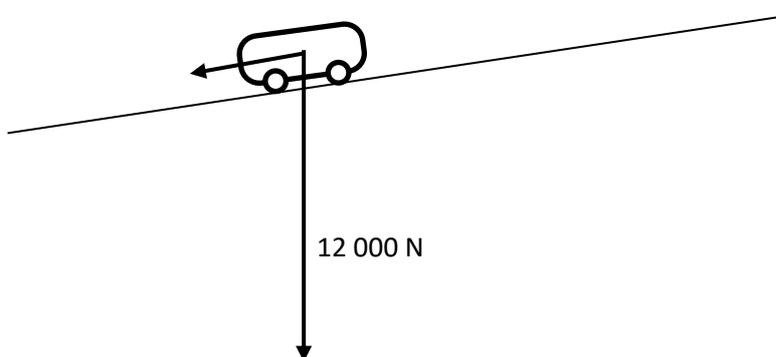


Clearly the component in the direction we want is $5 \cos 40^\circ$. In general, the component of any vector in any given direction will be the magnitude of the vector multiplied by the cosine of the angle between the two directions. If the directions are more than 90° apart, then the component will be negative.

Component of a force

Consider a car standing on a sloping road without the breaks on. The road slopes 8° , the mass of the car is 1200 kg and thus its weight is 1200×9.8 N, i.e. 11 760 N. The car might be expected to start rolling down the hill, i.e. to accelerate in a direction parallel to the road.

There is a component of the weight vector acting parallel to the road and this will cause the acceleration.



The magnitude of the component will be $11\,760 \cos 82^\circ$, i.e. 1637 N. As $F = ma$, the acceleration of the car will be $1637 \div 1200 = 1.36$ m/s². This of course assumes that there is no friction or other resistive force.

Practice

Q19 Find

Scalar Product (or Dot Product)

In Cartesian or Unit Vector Form

The scalar product (more commonly known as the dot product) is a binary operation with two vectors as input and one scalar (a number) as output. The dot product of \mathbf{a} and \mathbf{b} is written $\mathbf{a} \cdot \mathbf{b}$ and pronounced 'a dot b'.

It is called the scalar product or dot product to distinguish it from another binary operation, the vector product or cross product, which we will look at later.

The dot product is calculated by finding the product of the x -components, adding the product of the y -components and the product of the z -components (if the vectors are 3D).

For instance, $(\hat{i} + 3\hat{k}) \cdot (2\hat{k} - 4\hat{k}) = 1 \times 2 + 3 \times -4 = -10$

In Cartesian form this is $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = 1 \times 2 + 3 \times -4 = -10$

For 3D vectors, $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5.5 \\ 1 \\ 3 \end{pmatrix} = 2 \times 5.5 + -2 \times 1 + 0 \times 3 = 9$

In Polar Form

If we have the vectors in polar form, then the scalar product can be worked out by multiplying the magnitudes and then multiplying by the cosine of the angle between the two vectors, θ . This will always give the same result as multiplying in Cartesian form.

So $(2, 25^\circ) \cdot (5, 135^\circ) = |(2, 25^\circ)| \times |(5, 135^\circ)| \times \cos \theta = 2 \times 5 \times \cos 110^\circ = -3.4$

Vectors pointing in the same direction have a dot product equal to the product of their magnitudes (because $\cos 0^\circ = 1$).

Perpendicular vectors have a dot product of 0 (because $\cos 90^\circ = 0$).

Vectors pointing in opposite directions have a dot product of the negative of the product of their magnitudes (because $\cos 180^\circ = -1$).

Practice

Q20 Find

Applications of the Dot Product

The dot product might appear to be a fairly randomly defined operation without a lot of use, but it does actually have quite a few applications.

Calculating Work Done

If a force causes an object to be displaced, then the work done on the object by the force is equal to the scalar product of the force vector and the displacement vector. This should make sense as force and displacement are vectors with magnitudes measured in Newtons and metres respectively and work (or energy) is a scalar measured in Joules.

To see why this is the case consider that the magnitude of the component of the force in the direction of the displacement will be equal to the magnitude of the force vector multiplied by the cosine of the angle between the two vectors. The work done is the magnitude of the displacement multiplied by the magnitude of the force in that direction. This will be the same as the dot product of the two vectors.

Finding the angle between vectors

Suppose we have two vectors in polar form, say $(10, 30^\circ)$ and $(25, 70^\circ)$. It is easy to see that the angle θ between the vectors is 40° .

But this is not so easy if the vectors are in component form, e.g. $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

To do this we use the fact that the dot product can be expressed in polar form and in cartesian form and we equate the two with the only unknown being θ , the angle between them.

$$\mathbf{a} \cdot \mathbf{b} = 1 \times -2 + 3 \times 5 = 13$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = \sqrt{1^2 + 3^2} \sqrt{2^2 + 5^2} \cos \theta = 17.03 \cos \theta$$

$$\text{So } 13 = 17.03 \cos \theta, \quad \cos \theta = 0.763 \quad \text{and } \theta = 40.2^\circ.$$

Practice

Q21 Find

Finding the Distance between two Places on Earth

One useful application of using the dot product to find the angle between vectors is finding shortest distances between two points on the Earth, given their latitude and longitude. This is done as follows:

1. express the two places as 3D displacement vectors from the centre of the earth in polar form. For convenience use the radius of the Earth as the unit of measurement so that the magnitude of each vector is 1.
2. convert the vectors to cartesian form.
3. equate the two forms and calculate θ .
4. use the value of θ and the radius of the Earth (6370 km) to calculate the arc length between the two places.

Practice

Q22 Find the shortest distance by air between Mumbai, India ($19^\circ\text{N } 73^\circ\text{E}$) and Brisbane, Australia ($27^\circ\text{S } 153^\circ\text{E}$).

Q23 Find the shortest distance by sea between Dunedin, New Zealand ($46^\circ\text{S } 171^\circ\text{E}$) and Lima, Peru ($12^\circ\text{S } 77^\circ\text{W}$).

Vector Product (or Cross Product)

Another binary operation that is useful to define on vectors in 3 dimensions is the vector product. The vector product of \mathbf{a} and \mathbf{b} is written $\mathbf{a} \times \mathbf{b}$, pronounced 'a cross b', or sometimes in physics $\mathbf{a} \wedge \mathbf{b}$, pronounced 'a vec b'. We will use $\mathbf{a} \times \mathbf{b}$.

The vector product of two vectors is a vector. The magnitude of $\mathbf{a} \times \mathbf{b}$ is the product of the magnitudes of \mathbf{a} and \mathbf{b} , multiplied by the sine of the angle between them i.e. $|\mathbf{a}| |\mathbf{b}| \sin \theta$. The direction of $\mathbf{a} \times \mathbf{b}$ is perpendicular to both \mathbf{a} and \mathbf{b} . Of course this narrows it down to two directions 180° apart. To decide which, imagine being at the end of the vector arrow for \mathbf{a} (the vector written first in the product) looking towards the end of the vector arrow for \mathbf{b} (the vector written second in the product). Vector $\mathbf{a} \times \mathbf{b}$ will then point to your left.

Another way to think about this is to use the right-hand rule. Arrange the thumb and first two fingers of your right hand like this:

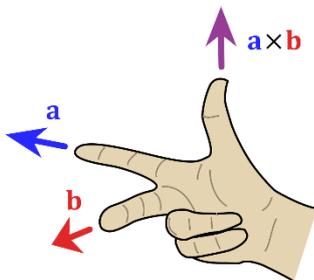


Image by Acdx - Self-made, based on
Image:Right_hand_cross_product.png, CC BY-SA 3.0,
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If your first finger represents the first vector and your second finger represents the second vector, then your thumb represents the cross product.

This of course means that the vector product operation is not commutative, i.e. $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$. In fact $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

Calculation of the Vector Product

If the magnitudes of the two vectors are known and the angle between them is known, then the calculation is straightforward. For example if \mathbf{a} has magnitude 5 and bearing 060° and \mathbf{b} has magnitude 6 and bearing 110° , then $\mathbf{a} \times \mathbf{b}$ has magnitude $5 \times 6 \times \sin 50^\circ$, i.e. 23.0 and direction vertically downwards.

However, if $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, the calculation is more involved.

Let \mathbf{a} be $\begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix}$ and \mathbf{b} be $\begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$. Then $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} y_a z_b - z_a y_b \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$

This looks difficult to remember, but there are patterns that make it easier: x is the only letter that doesn't appear in the first row, y is the only one that doesn't appear in the second row and z is the only one that doesn't appear in the third row; the rows cycle through x, y, z ; and each term has an a subscript followed by a b subscript.

Practice

Q24 Find

Applications of the Vector Product

Force on a wire

One application of the vector product is in finding the force vector on a wire that carries a current through a magnetic field.

$\mathbf{F} = \mathbf{I} \times \mathbf{B}$ where \mathbf{F} is the force on 1 m of the wire in Newtons, \mathbf{I} is the current vector in amps and \mathbf{B} is the magnetic field vector in Teslas. Note the use of upper-case letters for the vectors here is counter to the mathematical convention. It is done because \mathbf{F} , \mathbf{I} and \mathbf{B} are the conventional abbreviations for force current and magnetic field in physics.

Intersection of Two Planes

Practice

Q25 Find

Storage Vectors

A vector is any quantity that takes more than one number to specify it. For example, a vector in 2 dimensions takes 2 numbers to specify it; a vector in 3 dimensions takes 3 numbers.

The idea of vectors can be made more general by considering them as any quantity which requires more than one number to specify it. An example is the medal tallies for countries in the Olympics. The number of gold, silver and bronze medals won by a country in the Olympics would be a vector because it takes 3 numbers to specify it. For instance, Italy might have won 4 gold, 2 silver and 7 bronze to date. We could write this in vector form as $\begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$, provided of course that we have a convention that the order is 'gold, silver, bronze'.

Such vectors are called storage vectors. By contrast, vectors which have magnitude and direction are called geometric vectors.

Now suppose Poland's tally was $\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$. We might be interested in which country was

doing better. Of course these vectors can't be directly compared to see which is higher. But, if we could convert them to single numbers (scalars), then we could compare them.

To do this, we might decide that a gold is worth 3 points, a silver 2 points, and a bronze 1 point. We could then calculate the number of points for each country: Italy has $4 \times 3 + 2 \times 2 + 7 \times 1$, which comes to 23; Poland has $3 \times 3 + 6 \times 2 + 3 \times 1$, which comes to 24. So, by that way of reasoning, Poland is doing better.

We could write the points for each colour of medal as a vector like this: $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, again gold first, then silver then bronze. The scalar product of the tally vectors and the points vector will give us:

$$\text{For Italy: } \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 4 \times 3 + 2 \times 2 + 7 \times 1 = 23$$

$$\text{For Poland: } \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3 \times 3 + 6 \times 2 + 3 \times 1 = 24$$

As you can see, this is doing exactly the same calculation as before. The advantage of the vector method is that it can be easier when using a calculator or computer, particularly if we were working out the scores for many countries.

So both geometric vectors and storage vectors can be operated on mathematically.

Practice

Q26 Find

Solve

Revise

Revision Set 1

Q71 8 kg of potatoes

Answers
