M1 Maths

N6-1 Complex Numbers

- plotting complex numbers on an Argand diagram
- converting between Cartesian, polar and exponential form
- adding, subtracting, multiplying and dividing complex numbers and finding powers and roots
- application of complex numbers to trigonometric identities and fractals

Summary Learn Solve Revise Answers

This is just a very brief introduction to a major area of higher mathematics rather than a thorough and rigorous treatment. You may need to go further if you study specialised maths courses in upper high school which include complex numbers.

Summary

A pure imaginary number is a multiple of i. i is $\sqrt{-1}$. An imaginary number is the sum of a real number and a pure imaginary number, e.g. $2 + 3i$. The set of real numbers and imaginary numbers is called the complex numbers.

Complex numbers can be plotted on the Argand plane with the real part on the horizontal axis and the imaginary part on the vertical axis.

Mathematical operations can be performed on complex numbers and they can be applied to many practical situations affording methods which are quicker and easier than methods involving only real numbers.

Learn

Introduction

How do you write 0 in Roman numerals? Answer: you can't; the Romans didn't have a symbol for 0. They didn't use the number in their mathematics.

In about the 9^{th} Century, the Indians and Arabs developed the place-value system for numbers which made calculation very much simpler. (Try dividing MDCCLXIV by XXVII without using modern numbers.)

The place-value system needed a zero to express numbers like 109. So zero arrived with place value.

Place value numbers and zero were treated with suspicion in Europe and it took several more centuries before they were fully accepted. We still use Roman numerals for movie dates and on some clocks.

It wasn't until the 11th Century that anyone conceived of the idea of a negative number. Negative numbers made some calculations easier, especially in the area of algebra and relations. But these were treated with even more suspicion than place-value numbers. After all, you could have 3 apples; you could even have 0 apples; but you couldn't have −2 apples.

People said that negative numbers don't exist. But others countered 'neither do positive ones – you can't put your finger on 2 any more than you can put your finger on −2. You can only put your finger on apples. Negative numbers are a concept that makes some maths easier and still gives the right answers. So we use them.

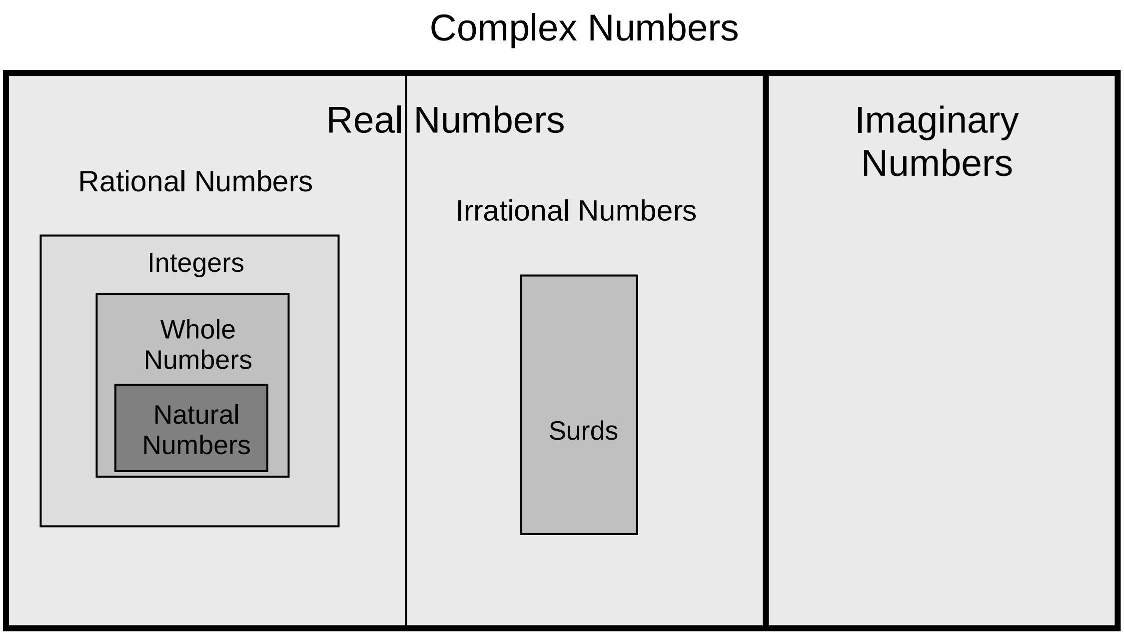
In the 19th Century, mathematicians invented the idea of imaginary numbers. These involve $\sqrt{-1}$. You will know that no real number squared makes −1. So in that sense, $\sqrt{-1}$ doesn't exist. But using it can make some calculations a lot quicker. So we do.

To save time writing it, we call $\sqrt{-1}$ i. We use i because it sands for imaginary. Square roots of negative numbers were said to be imaginary, but they are no more imaginary than negative numbers themselves. However, the name stuck and now we call any number containing i imaginary.

So there are real numbers and imaginary numbers. We call the set of all these numbers Complex Numbers. Again, this is a somewhat unfortunate name because 2 is then a complex number. But it has stuck.

If we call the set of real numbers R, the set of imaginary numbers M and the set of complex numbers C, then $C = R \cup M$. Though you can't make Pina colada from them!

Our number set Venn diagram will now look like this.



The good news is that there are no more sets of numbers to add. All modern mathematics is done with complex numbers (real and imaginary).

Imaginary numbers and pure imaginary numbers

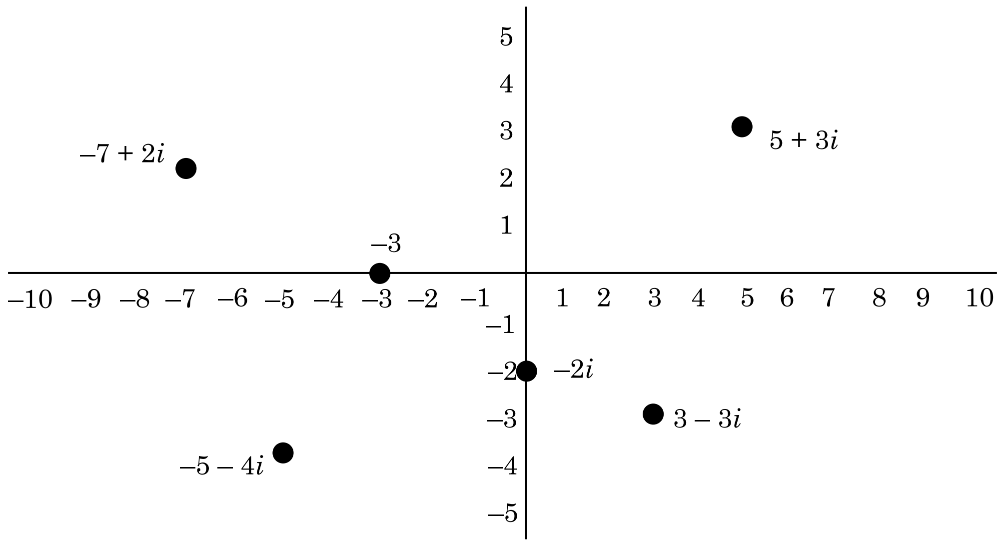
i is $\sqrt{-1}$, but we can take the square root of any negative number. For example, $\sqrt{-16} = \sqrt{16}\,\sqrt{-1} = 4\sqrt{-1} = 4i$.

You can add an imaginary number to a real number, e.g. to get $2 + 3i$. This contains $\sqrt{-1}$ and so is also imaginary.

To distinguish numbers like $2 + 3i$, which has a real part and an imaginary part, from numbers like $4i$, which only have an imaginary part, we call the latter pure imaginary numbers. Both are imaginary, but $4i$ is purely imaginary.

Argand Diagrams

We can plot real numbers on a 1-dimensional number line. For complex numbers, we need a 2-dimensional plane called the Argand Plane. We plot the real part of the number on the horizontal axis and the imaginary part on the vertical axis.



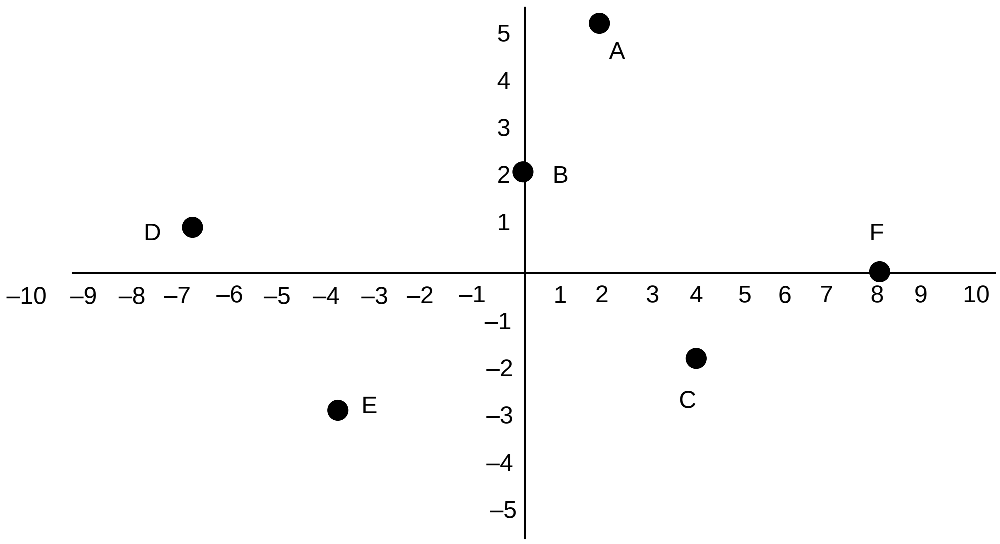
The horizontal axis contains all the real numbers; the vertical axis contains all the pure imaginary numbers. The quadrants contain other imaginary numbers.

Note that when we talk about the real and imaginary parts of a complex number like $-5 - 4i$, we say that the real part is –5 and that the imaginary part is –4, not $-4i$. The real and imaginary parts are the coordinates on the Argand diagram.

We can also call the real and imaginary parts of $-7 + 2i$ Re($-7 + 2i$) and Im($-7 + 2i$). So Re($-7 + 2i$) = –7 and Im($-7 + 2i$) = 2.

Practice

Q1 Give the numbers at the points A to F on the diagram below.



Q2 Copy the axes from P1 and plot the following numbers on them:

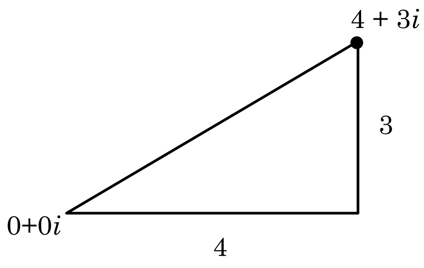
(a) G $3 + i$ (b) H $(-4 + 4i)$ (c) I $(-2 - 5i)$

(d) J $(-i)$ (e) K 0 (f) L $3 - 4i$

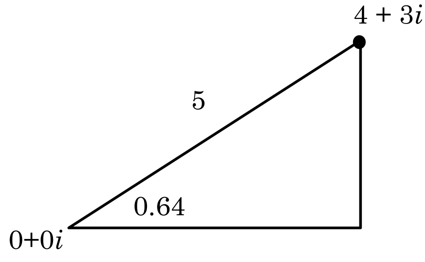
Polar Form

In Module G1-3 (Position) we specified position in terms of coordinates and in terms of distance and bearing (direction). So far we have considered complex numbers in terms of coordinates, but we can use distance and direction.

Consider the number $4 + 3i$. Relative to the origin of the Argand plane, it is 4 to the right and 3 up. We call this way of writing it Cartesian form because we are essentially using Cartesian coordinates.



But we can write the number another way – in polar form. Using Pythagoras, we can say that it is 5 units from the origin and, using trigonometry, we can say that it is in a direction 0.64 radians anti-clockwise from the positive real axis.



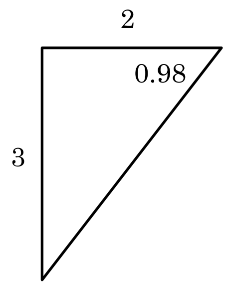
We can then see that the real part of the number is 5 cos 0.64 and the imaginary part is 5 sin 0.64.

So the number can be written as 5 cos 0.64 + 5 sin 0.64 i.

More normally, we write this as 5(cos 0.64 + i sin 0.64). And often we abbreviate it to 5 cis 0.64, where cis is short for 'cos + i sin'.

Cartesian form uses the real and imaginary parts and is written in the form $x + yi$; polar form uses the distance and angle and it written in the form r cis θ.

You have seen above how to convert Cartesian form to polar form using Pythagoras and trigonometry. If the number is not in the first quadrant, we still use the angle anti-clockwise from the positive real axis, but it is worth doing a quick sketch to see where the number is so that we get θ right. E.g. for $-2 - 3i$, $\theta = \pi + 0.98 = 4.12$.



Converting the other way is trivial: we just have to find cos 0.64 and sin 0.64 on the calculator and multiply them both by 5.
For example 3 cis 1.8 = 3 cos 1.8 + 3 sin 1.8 $i = -0.68 + 2.92i$.

For angles which are simple fractions of π, it is conventional to use these fractions rather than decimals.

Practice

Q3 Write these numbers in polar form:

(a) $12 + 5i$	(b) $5 + 2i$	(c) $-2 + i$	(d) $3 - 5i$
(e) $-1 - 2i$	(f) $3i$	(g) 5	(h) -2
(i) $4 + 4i$	(j) $-8i$	(k) $1 - \sqrt{3}\,i$	(l) $-\sqrt{2} + \sqrt{2}\,i$

Q4 Write these numbers in Cartesian form:

(a) 5 cis 0.4	(b) 2 cis $\pi/3$	(c) cis $\pi/2$	(d) 3.2 cis 2.44
(e) 4 cis 5	(f) 3 cis $7\pi/4$	(g) 2 cis 0	(h) cis π

Exponential Form

The third and last way of writing a complex number is in exponential form $re^{i\theta}$. Converting between polar and exponential form is easy because both forms are written in terms of r and θ. r cis $\theta = re^{i\theta}$.

So 4 cis $\pi/6 = 4e^{i\pi/6}$ and $-2e^{5i} = -2$ cis $5i$.

r cis $\theta = re^{i\theta}$ is known as Euler's Formula. Proving it requires quite a lot of work involving the use of calculus to expand e^x, sin x and cos x as MacLaurin series and will not be done here. You do know enough maths now to be able to follow it, though. You might like to chase it up with an Internet search.

One interesting consequence of Euler's Formula arises if we sub 1 for r and π for θ. We get

$$e^{i\pi} + 1 = 0$$

This is known as Euler's Identity and is considered by many as the most beautiful equation in mathematics, as celebrated as $E = mc^2$ is by physicists. The beauty lies in the fact that it contains the five most fundamental numbers of mathematics in one very concise equation.

Practice

Q5 Write these numbers in exponential form:

(a) $12 + 5i$ (b) $5 + 2i$ (c) $-2 + i$ (d) $3 - 5i$

(e) $-1 - 2i$ (f) $3i$ (g) 5 (h) -2

(i) $4 + 4i$ (j) $-8i$ (k) $1 - \sqrt{3}\, i$ (l) $-\sqrt{2} + \sqrt{2}\, i$

Q6 Write these numbers in polar form:

(a) $2e^{3i}$ (b) $4e^{\pi i}$ (c) $3.5e^{3\pi/2\, i}$

Q7 Write these numbers in Cartesian form:

(a) $5e^{2i}$ (b) $e^{\pi/3\, i}$ (c) $2e^{\pi i}$

Adding and subtracting complex numbers

Adding and subtracting complex numbers is easy in Cartesian form, difficult in the other forms, so always convert to Cartesian form first if necessary.

All we do is add or subtract the real parts to get the real part of the result and add or subtract the imaginary parts to get the imaginary part of the result.

For example, $4 + 3i + 2 - i = 6 + 2i$ $\quad -2 + i - (2 - 4i) = -4 + 5i$

Practice

Q8 Perform these operations, giving the answer in the same form as the question:

(a) $(8 + 5i) + (3 - 2i)$ (b) $(-5 + 2i) - (2 - i)$ (c) $(-2 + i) + (2 - 3i)$

(d) 4 cis 2.8 + 2 cis 1.2 (e) cis $\pi/4$ – 3 cis $3\pi/4$

(f) $2e^{3i} + 3e^{i}$ (g) $5e^{0.5\pi i} - 2e^{1.75\pi i}$

Multiplying and dividing complex numbers

Multiplying and dividing complex numbers can be done in Cartesian form, but it is easier in polar or exponential form. If the numbers are in Cartesian you can choose whether or not to convert.

Doing it in Cartesian form

To multiply $3 + 2i$ by $4 - i$, we just expand brackets like this:

$$(3 + 2i)(4 - i)$$
$$= 12 - 3i + 8i - 2i^2$$
$$= 12 + 5i + 2 \text{ (because } i^2 = -1)$$
$$= 14 + 5i$$

To divide $3 + 2i$ by $4 - 3i$, we write the expression as a fraction, then multiply the top and bottom by the conjugate of the denominator.

$$\frac{3+2i}{4-3i}$$
$$= \frac{3+2i}{4-3i} \times \frac{4+3i}{4+3i}$$
$$= \frac{(3+2i)(4+3i)}{(4-3i)(4+3i)}$$
$$= \frac{6+17i}{25}$$
$$= \frac{6}{25} + \frac{17}{25}i$$

Note that we should five the answer in standard Cartesian form, which is $x + yi$.

Doing it in polar or exponential form

First a couple of new words. In polar form r cis θ or exponential form $re^{i\theta}$, r is called the modulus of the number and θ is called the argument of the number. So for the number 3 cis 2, the modulus is 3 and the argument is 2.

To multiply two complex numbers, multiply the moduli and add the arguments.
So 3 cis 2 × 5 cis 4 = 15 cis 8 $6e^{4i} \times 2e^{-3i} = 12e^{i}$.

To divide two complex numbers, divide the moduli and subtract the arguments.
So 3 cis 2 ÷ 5 cis 4 = 0.6 cis (−2) $6e^{4i} \div 2e^{-3i} = 3e^{7i}$.

Again, it won't be shown why this works. You will just have to accept it for now.

Practice

Q9 Perform these operations, giving the answer in the same form as the question:

(a) $4 \text{ cis } 5 \times 3 \text{ cis } 1.5$ (b) $\text{cis } {}^{\pi}/_{4} \div 2 \text{ cis } {}^{3\pi}/_{4}$

(c) $2e^{3i} \times 3e^{i}$ (d) $5e^{0.5\pi i} \div 2e^{1.75\pi i}$

(e) $(8 + 5i) \times (3 - 2i)$ (f) $(-5 + 2i) \div (2 - i)$

Powers of complex numbers

Finding a power of a complex number is generally easier in polar or exponential form. It is just an extension of multiplying.
For example, $(5 \text{ cis } 2)^3 = 5 \text{ cis } 2 \times 5 \text{ cis } 2 \times 5 \text{ cis } 2 = 125 \text{ cis } 6$.

So $(r \text{ cis } \theta)^n = r^n \text{ cis } n\theta$. In the same way $(re^{i\theta})^n = r^n e^{in\theta}$.

This rule applies for all rational indices. So $(5 \text{ cis } 2)^{-3.5} = 5^{-3.5} \text{ cis } -7$.

Practice

Q10 Perform these operations, giving the answer in the same form as the question:

(a) $(2 \text{ cis } 5)^3$ (b) $(\text{cis } {}^{\pi}/_{4})^6$

(c) $(3e^{6i})^{-1.5}$ (d) $\sqrt{9e^{8i}}$

(e) $(2 + i)^3$ (f) $\sqrt[4]{5 - 3i}$

Applications of complex numbers

Complex numbers have applications in many area, in particular in solving more complex differential equations, AC electronics and fluid flow. Unfortunately, all these are beyond high school maths and are studied at university level.

However, we will have a brief look at a couple of applications here so that you can see that they do have applications to situations that involve only real numbers.

Proving Trigonometric identities

Complex numbers can be used to prove many trigonometric identities. Here is one example: proving that $\cos 2x = \cos^2 x - \sin^2 x$.

Let z be a complex number. (We usually use z for variables which can be imaginary.)

$z = r(\cos\theta + i\sin\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

Let the modulus of z be 1.

Then $z^n = \cos n\theta + i\sin n\theta$ *and* $z^{-n} = \cos n\theta - i\sin n\theta$

$z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$

$\cos n\theta = \frac{z^n + z^{-n}}{2}$ and $\sin n\theta = \frac{z^n - z^{-n}}{2i}$

$\cos\theta = \frac{z + z^{-1}}{2}$ and $\sin\theta = \frac{z - z^{-1}}{2i}$

Note: normally this would be an established result that we would take as the starting point for the proof. So the proof would normally be just the part below this line.

RTP: $\cos 2x = \cos^2 x - \sin^2 x$ [RTP stands for 'Required to prove']

RHS $= \left(\frac{z+z^{-1}}{2}\right)^2 - \left(\frac{z-z^{-1}}{2}\right)^2$ [RHS stands for 'Right hand side']

$= \left(\frac{z^2+2+z^{-2}}{4}\right)^2 - \left(\frac{z^2-2+z^{-2}}{-4}\right)^2$

$= {}^1/_4\,(z^2 + 2 + z^{-2} + z^2 - 2 + z^{-2})$

$= {}^1/_4\,(2z^2 + 2z^{-2})$

$= \frac{z^2+z^{-2}}{2}$

$= \cos 2\theta$ = LHS [LHS stands for 'Left hand side']

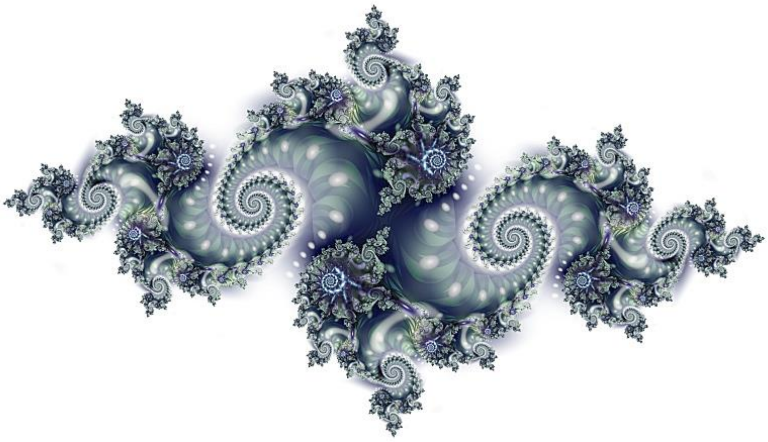
QED. [QED stands for Quod erat demonstrandum, Latin for 'What was to be proved.]

Fractals

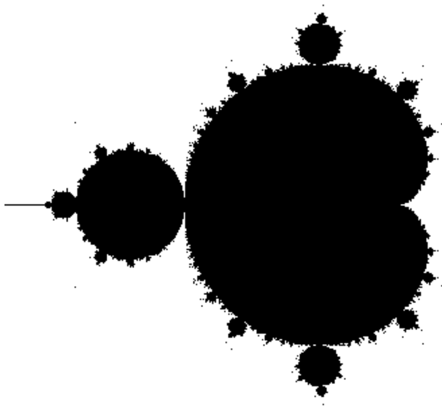
Fractals are shapes whose boundaries are infinitely convoluted. You can zoom in on the boundary for ever and it never becomes smooth. You might like to go to YouTube and search Mandelbrot zoom to watch some examples of zooming in to the boundary of the Mandelbrot set. The images can be quite beautiful.

Complex numbers are used to generate fractals like Julia sets and the Mandelbrot set.

This is a Julia set:



This is the Mandelbrot set:



Fractals like these were envisaged in the early 20th Century, but, because of the huge number of calculations that had to be performed, couldn't actually be constructed until the advent of electronic computers.

How to construct a Julia set

Pick a complex number, c. The value you choose will determine the shape of the Julia set. The image above is for $c = -0.6 + 0.3i$.

Then pick another complex number, $z_1 = 0.8 + 0.1i$. Plot this on the Argand plane.

Then calculate $z_2 = z_1^2 + c$ and plot z_2 on the diagram. Join z_1 to z_2 with a line.

Calculate $z_3 = z_2^2 + c$ and plot z_3. Join z_2 to z_3 with a line.

Calculate $z_4 = z_3^2 + c$ and plot z_4. Join z_3 to z_4 with a line.

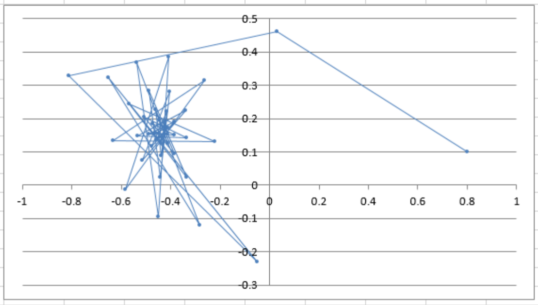
Calculate $z_5 = z_4^2 + c$ and plot z_5. Join z_4 to z_5 with a line.

And so on.

This can be done using a spreadsheet like this.

n	$Re(z_n)$	$Im(z_n)$
1	0.8	0.1
2	0.03	0.46
3	-0.8107	0.3276
4	-0.05009	-0.23117
5	-0.65093	0.323157
6	-0.28072	-0.12071
7	-0.53577	0.367769
8	-0.44821	-0.09408
9	-0.40796	0.384332
10	-0.58128	-0.01358
11	-0.2623	0.315792
12	-0.63092	0.134337
13	-0.21998	0.130488
14	-0.56864	0.242591
15	-0.3355	0.024109
16	-0.48802	0.283823
17	-0.44239	0.022978
18	-0.40482	0.279669
19	-0.51434	0.073571

Re(c)	Im(c)
-0.6	0.3



What we find is that z gets closer and closer to one point on the diagram – roughly $-0.45 + 0.15i$.

If we do the same starting with $z_1 = 1 + 0.1i$, we find that, instead of getting closer and closer to one point, z gets further and further away from the origin.

We say that $0.8 + 0.1i$ is part of the prisoner set, $1 + 0.1i$ is part of the escape set. Every complex number z will be part of the prisoner set or part of the escape set.

Now, if we colour the points on the Argand plane which are part of the prisoner set black and those which are part of the escape set white, we will get a black shape surrounded by white. The shape of the black shape will depend on the value chosen for c. For some values of c it is fairly simple or even non-existent. But for other values, it will be very complex with a very contorted boundary – a fractal. $c = –0.6 + 0.3i$ is one of the best examples with a very interesting, pleasing and symmetrical shape.

How to construct the Mandelbrot set

For some values of c, the Julia set is connected, i.e. in one piece, and includes the origin; for other values of c, the Julia set is disconnected, i.e. in many separate pieces, and does not include the origin. The set of values of c for which the Julia set is connected is the Mandelbrot set.

Gradational shading

You will notice that the picture of the Julia set above is not just black and white, but shades of grey. The escape set is shaded. The shading is determined by the number of iterations of z required before it goes outside a circle of radius 2 centred on the origin. If the number of iterations is large (e.g. if z_{50} is the first to go outside), then the point c is shaded dark; if the number of iterations is small (e.g. is z_5 goes outside), then the point is shaded lighter. Shading can be done in different colours too to produce coloured images.

The Mandelbrot set is often shaded too. The Mandelbrot set can be thought of as the set of c values for which $z_1 = 0 + 0i$ escapes. When iterated. Again, the number of iterations determine the shading or colour. If you watched a Mandelbrot zoom, you would probably have noticed the colours. When you have zoomed in a long way and are extremely close to the boundary, the colours may represent many thousands of iterations. It is clear why computers are needed to generate detained Julia and Mandelbrot sets.

Chaos Theory and Butterflies

Fractals are intimately related to chaos theory. Chaos theory is the study of system for which a tiny change in input can produce major changes in output. In the Julia set, a change to z1 of one part in a billion can lead to z changing from being in the prisoner set and remaining near the origin to z being in the escape set and going off to infinity.

This is an example of the butterfly effect where the tiniest change to the atmosphere now can lead to huge changes down the track. For instance, whether butterfly flaps its wings in Japan could make the difference between Florida being hit by a hurricane or not being hit by a hurricane a couple of years down the track.

Solve

Q51 If you feel so inspired, use a spreadsheet to plot the iterations of a few complex numbers z according to the rule $z_2 = z_1^2 + c$. Try different values for c. Try other rules like $z_2 = z_1^3 + c$.

Q52 Download a fractal generating program and play with it. There are free ones available on the Internet. Fractint is quite a good one.

Revise

Q61 Write these complex numbers in all three forms – Cartesian, polar and exponential:

(a) $2 - 3i$ (b) $4 \text{ cis } {}^{5\pi}/_{6}$ (c) $2e^{-0.2i}$

Q62 Perform the following operations giving the answer in the same form as the question:

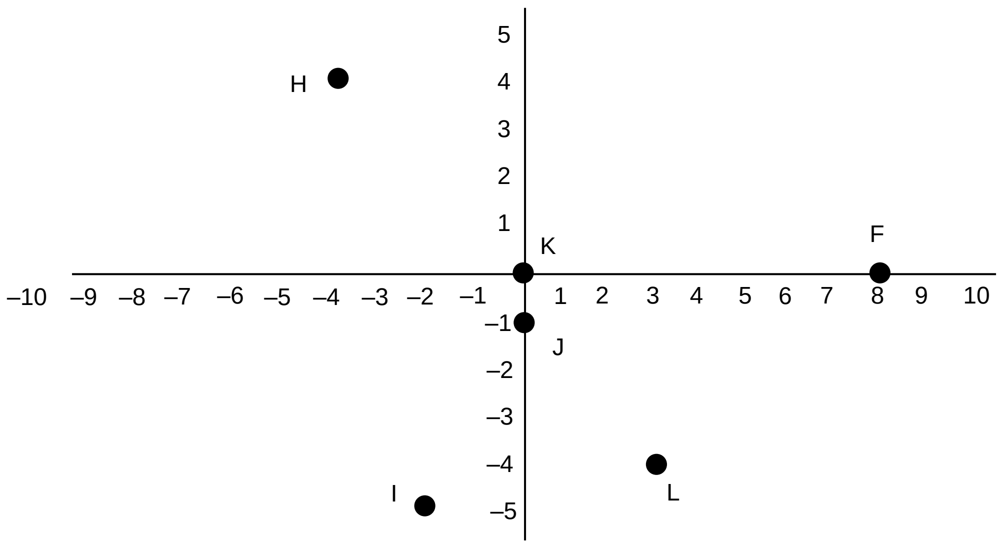
(a) $(4 - i) - (3 + 3i)$ (b) $(2 + 3i) \div (4 - 2i)$ (c) $4 \text{ cis } {}^{\pi}/_{4} \times 2 \text{ cis } {}^{3\pi}/_{4}$

(d) $2e^{2i} + 3e^{-3i}$ (e) $(4 \text{ cis } {}^{2\pi}/_{3})^5$ (f) $\sqrt[3]{3 - 4i}$

Answers

Q1 (a) $2 + 5i$ (b) $2i$ (c) $4 - 2i$ (d) $-7 + i$ (e) $-4 - 3i$ (f) 8

Q2



Q3	(a) 13 cis 0.39	(b) 5.39 cis 0.38	(c) 2.24 cis 2.68	(d) 5.83 cis 5.25
	(e) 2.24 cis 4.25	(f) 3 cis $\pi/2$	(g) 5 cis 0	(h) 2 cis π
	(i) 5.66 cis $\pi/4$	(j) 8 cis $3\pi/2$	(k) 2 cis $5\pi/3$	(l) 2 cis $3\pi/4$
Q4	(a) $4.61 + 1.95i$	(b) $1 + \sqrt{3}i$	(c) i	(d) $-2.44 + 2.07i$
	(e) $1.13 - 3.84i$	(f) $3\sqrt{2} - 3\sqrt{2}i$	(g) 2	(h) -1
Q5	(a) $13\ e^{0.39i}$	(b) $5.39\ e^{0.38i}$	(c) $2.24\ e^{2.68i}$	(d) $5.83\ e^{5.25i}$
	(e) $2.24\ e^{4.25i}$	(f) $3\ e^{\pi/2 i}$	(g) $5\ e^{0}$	(h) $2\ e^{\pi i}$
	(i) $5.66\ e^{\pi/4 i}$	(j) $8\ e^{3\pi/2 i}$	(k) $2\ e^{5\pi/3 i}$	(l) $2\ e^{3\pi/4 i}$
Q6	(a) 2 cis 3	(b) 4 cis π	(c) 3.5 cis $3\pi/2$	
Q7	(a) $-2.08 + 4.55i$	(b) $\frac{1}{2} + \sqrt{3}i$	(c) -2	
Q8	(a) $11 + 3i$	(b) $-7 + 3i$	(c) $-2i$	
	(d) $-3.04 + 3.20i$	(e) $4\sqrt{2} - 2\sqrt{2}i$		
	(f) $-0.36 + 2.81i$	(g) $-2\sqrt{2} + (5 + 2\sqrt{2})i$		
Q9	(a) 12 cis 6.5	(b) ½ cis $3\pi/2$		
	(c) $6e^{4i}$	(d) $2.5e^{-1.25\pi i}$		
	(e) $34 - i$	(f) $-12/5 - i/5$		
Q10	(a) 8 cis 15	(b) 6 cis $3\pi/2$		
	(c) $\frac{1}{3\sqrt{3}}e^{-9i}$	(d) $3e^{4i}$		
	(e) $2 + 11i$	(f) $0.21 + 1.54i$		
Q61	(a) $2 - 3i$	$\sqrt{13}$ cis 5.30	$\sqrt{13}e^{5.30i}$	
	(b) $-2 + 2\sqrt{3}i$	4 cis $5\pi/6$	$4e^{5\pi/6}$	
	(c) $1.96 - 3.40i$	2 cis -0.2	$2e^{-0.2i}$	
Q62	(a) $1 - 4i$	(b) $1/10 + 4/5 i$	(c) 8 cis π	
	(d) $-3.80 + 1.40i$	(e) 1024 cis $4\pi/3$	(f) 1.71 cis 1.79	