M1 Maths

M6-1 Radians

- radians, conversion between radians and degrees and the radian values of common angles

Summary Learn Solve Revise Answers

Summary

Radians are a measure of angle, more natural than degrees and more convenient in contexts like calculus, complex numbers etc.

The number of radians in an angle is equal to the arc length produced by the angle on a unit circle.

π radians = 180°. So to convert radians to degrees, divide by π, then multiply by 180. To convert degrees to radians, divide by 180, then multiply by π.

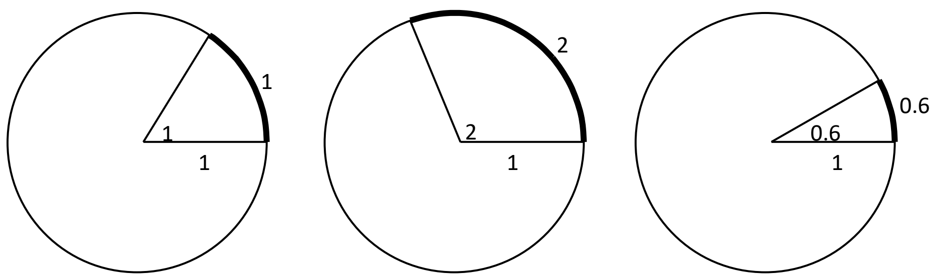
To work in radians on a calculator, set it to radian mode.

Learn

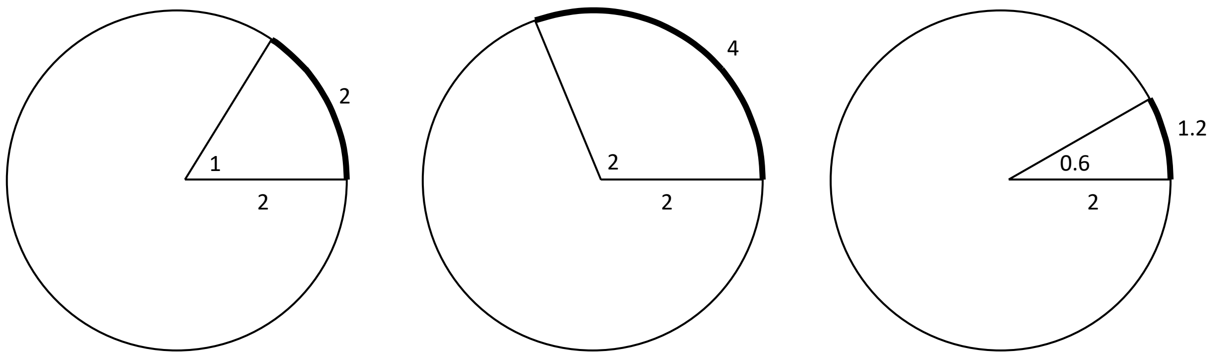
Later, we will learn how to differentiate $y = \sin x$ and $y = \cos x$. Differentiating these is easier, however, if we measure the angles involved in radians rather than in degrees. When doing any calculus on trigonometric functions, mathematicians use radians and, in fact, radians are used in many other situations in advanced mathematics, for instance complex numbers.

Consider angles drawn in a unit circle. Each angle produces a certain arc length on the circumference of the circle. The bigger the angle, the bigger the arc length.

We can use the arc length as a measure of the angle. If the arc length is 1, then the angle is defined as 1 radian; if it is 2, the angle is 2 radians; if it is 0.6, the angle is 0.6 radians and so on. So the angle in radians is equal to the arc length produced on a unit circle.



This is a more natural and less arbitrary unit of angle than the degree.
If the radius of the circle was 2 instead of 1, then an angle of 1 radian would have an arc length of 2. And so on.



In general, 1 radian is the angle for which the arc length is equal to the radius. It is about 57°.

The arc length will always be the radius multiplied by the number of radians, i.e. $a = r\theta$ where a is the arc length, r is the radius and θ is the angle measured in radians.

$a = r\theta$ is an easier relation than $a = \frac{2\pi r\theta}{360}$ which we have if θ is in degrees.

Just like ° is the symbol for degrees, ^{c} is the symbol for radians. So 2 radians can be written as 2^c. However, the symbol is rarely used. We usually write 2 radians as just plain 2. This doesn't generally lead to confusion.

Converting between degrees and radians

An angle of 360° on a unit circle produces an arc length of the whole circumference of the circle, i.e. $2\pi \times 1$, i.e. 2π. Therefore, $360° = 2\pi$ radians.

This allows us to convert between degrees and radians. It is actually generally easier to use $180° = \pi$. Think of this as 1 semicircle = π

So to convert 80° to radians, we first convert it to semicircles, i.e. $^{80}/_{180}$ or $^{4}/_{9}$, then change to radians, i.e. $^{4}/_{9}\pi$ or $\frac{4\pi}{9}$.

Radian measures are often given in terms of π. This allows the measures to be exact.

To convert say $\frac{3\pi}{2}$ to degrees, we express it as $^{3}/_{2}$ semicircles, and then as $^{3}/_{2} \times 180°$, which is 270°.

Practice

Q1 Convert the following angles to radians. Give the answers as decimals.

(a) 180° (b) 90° (c) 40° (d) 32°

(e) 100° (f) 50° (g) 225° (h) 7°

(i) 540° (j) 140° (k) 18.3° (l) 12.7°

(m) 72° (n) 123° (o) 36° (p) 510°

Q2 Convert the following angles to radians. Give the answers as fractions of π in simplest form

(a) 180° (b) 90° (c) 60° (d) –30°

(e) 120° (f) 45° (g) 270° (h) 360°

(i) 540° (j) 150° (k) –18° (l) 12°

(m) 72° (n) 123° (o) 36° (p) –510°

Q3 Convert the following angles to degrees.

(a) π (b) 2π (c) $\pi/4$ (d) $3\pi/4$

(e) $\pi/6$ (f) $\pi/9$ (g) $5\pi/6$ (h) 6π

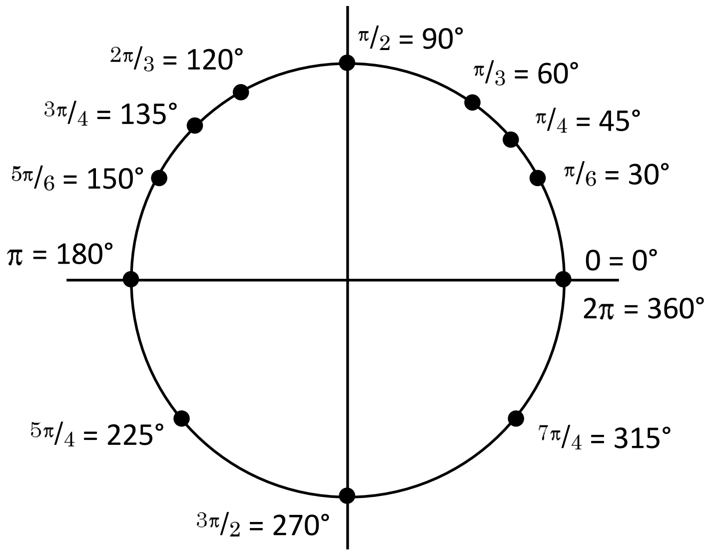
(i) $5\pi/2$ (j) $4\pi/3$ (k) $13\pi/4$ (l) 11π

(m) $5\pi/12$ (n) $3\pi/2$ (o) $-\pi/2$ (p) $-3\pi/4$

(q) 2.7 (r) 2 (s) 4.64 (t) –12.1

Common conversions to know

There are some angles for which it is worth knowing the radian measure as well as the degree measure. These are shown on this circle diagram. Try to learn them. Test yourself by reproducing this diagram from memory.



Using radians on your calculator

You probably realised long ago that calculators can be set to work in degrees or radians. On many calculators, radians is the default mode. You need to be able to change between the two modes. On a Casio graphics calculator, go into RUN mode, then press SHIFT, SET UP. Scroll down to Angle, then use F1 or F2 to choose degrees or radians.

A quick way of checking which mode you are in is to enter tan 45. If you get 1, you are in degrees. If you get 1.619775191.., you are in radians.

Practice

Q4 Set your calculator to radians to find the following.

(a) sin 0.6	(b) cos 1.1	(c) tan ¾	(d) sin 0
(e) sin $^{\pi}/_{2}$	(f) cos $^{\pi}/_{3}$	(g) tan $^{\pi}/_{5}$	(h) sin $^{\pi}/_{8}$
(i) cos $^{3\pi}/_{4}$	(j) tan $^{7\pi}/_{2}$	(k) $\sin^{-1}$ 0.5	(l) $\cos^{-1}$ 0.5
(m) $\tan^{-1}$ 1	(n) $\sin^{-1}$ 0.8	(o) $\cos^{-1}$ –0.3	(p) $\sin^{-1}$ –0.5

Trig Functions Using Radians

In Module A5-11 (Trig Functions) you graphed trig functions and found formulae for sinusoidal graphs where the independent variable was in degrees. You need to be able to do that in radians too.

At first, you might find it easier to think in degrees, then convert your final product to radians. However, you should aim to be able eventually to think in radians.

Note that, when working in radians, a is still the amplitude, d is still the mean value and c is still the negative of the phase shift. However, $period = 2\pi/b$ and $b = 2\pi/period$.

Practice

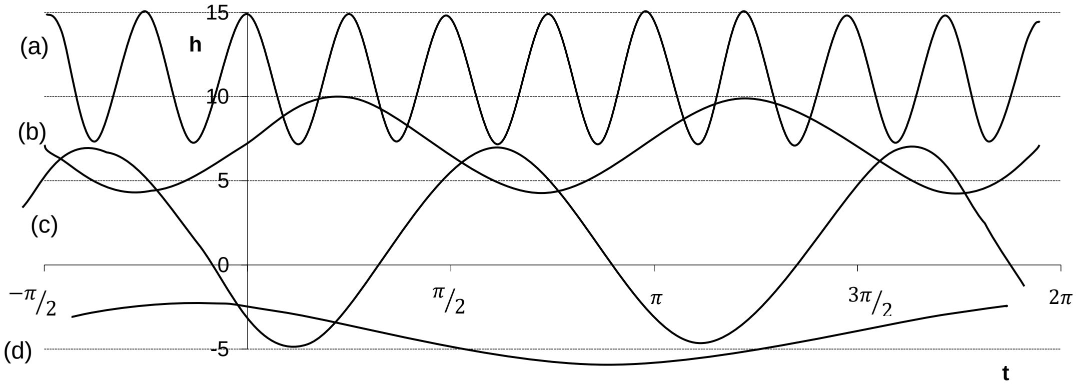
Q5 For each of the following functions,

(i) give the mean value, the amplitude, the period and the phase shift,

(ii) then find the values of the parameters a, b, c and d,

(iii) then find the formula.

Note that the phase shift will be approximate and will depend on the type of function used: sin, cos, −sin, −cos. Just one possibility is given in the answers, the one that has the smallest phase shift.

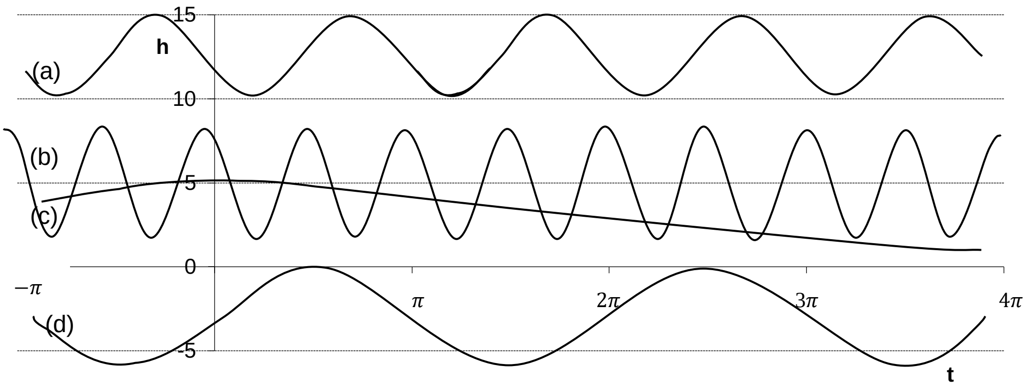


Q6 For each of these functions,

(i) give the mean value, the amplitude, the period and the phase shift,

(ii) then find the values of the parameters a, b, c and d,

(iii) then find the formula.



Q7 Sketch the graphs of the following functions over the domain $[-\pi, 2\pi]$. Check your sketches with your graphics calculator.

(a) $y = \sin x$

(b) $y = 2 \cos x$

(c) $y = -2 \sin x + 2$

(d) $y = 3 \cos 2x - 2$

(e) $y = \cos (x + \pi/6) + 2$

(f) $y = 2 \sin 2(x - \pi/12) - 3$

(g) $y = -0.5 \cos 3x - 1$

(h) $y = 1.5 \sin 2x - 3$

Trig Equations Using Radians

In Module A5-12 (Trig Equations) you solved trigonometric equations where the unknown was in degrees. You need to be able to do that in radians too. The only step that's different is the unit circle step.

Practice

Q8 Solve the following trigonometric equations for $0 \le x \le 2\pi$. Don't forget there will generally be more than one solution.

(a) $\sin x = 0.5$

(b) $4 \cos x = 0.3$

(c) $\sin (x + \pi/4) = 0.4$

(d) $3 \sin (x - \pi/6) + 2 = 2.9$

(e) $3 \cos 2x = 1.5$

(f) $\tan x = -1$

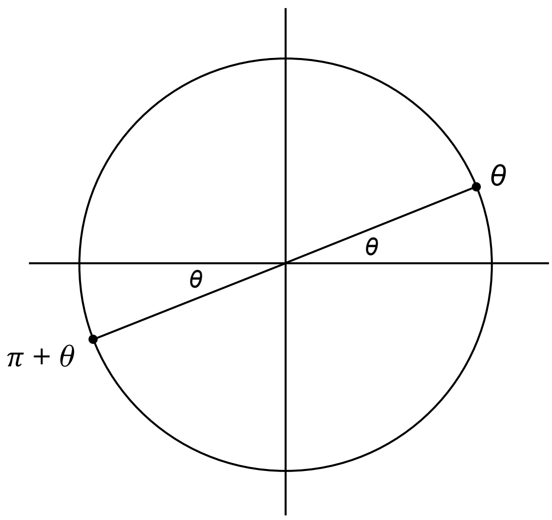
(g) $2 \tan (x - \pi/2) = 2\sqrt{3}$

(h) $2 \cos x = 5 \sin x$

Simplifying Trigonometric Expressions

The expression $\sin (\pi + \theta)$ can be simplified to $-\sin \theta$. This can be seen on the unit circle.

If we draw θ as a small angle in the first quadrant, then draw $\pi + \theta$, we can see that the sin of the latter is the negative of the sin of the former.



Other expressions can be similarly simplified. For example, $\cos (2\pi - \theta)$ can be simplified to $\cos \theta$.

Practice

Q9 Simplify the following expressions using the unit circle.

(a) $\sin (\pi - \theta)$

(b) $\cos (\pi - \theta)$

(c) $\sin (2\pi + \theta)$

(d) $\tan (\pi - \theta)$

(e) $\cos (2\pi - \theta)$

(f) $\tan (\pi + \theta)$

(g) $\sin (\pi + \theta)$

(h) $\cos (\pi + \theta)$

Q10 Simplify the following expressions using the unit circle and the trig identities you met in Module M5-1 (Unit Circle).

(a) $\sin (\pi/2 - \theta)$ (b) $\cos (\pi/2 + \theta)$

(c) $\sin (3\pi/2 - \theta)$ (d) $\cos (3\pi/2 + \theta)$

(e) $\sin (3\pi/2 + \theta)$ (f) $\cos (5\pi/2 - \theta)$

(g) $\sin^2 (\pi/2 - \theta) + \cos^2 (\pi/2 - \theta)$ (h) $\frac{\sin(\pi+\theta)}{\cos(\pi+\theta)}$

(i) $\frac{\sin(2\pi-\theta)}{\tan(2\pi-\theta)}$ (j) $\frac{\sin(2\pi+\theta)}{\tan(\pi-\theta)}$

Solve

Q51 You might have noticed along with DEG and RAD as angle modes on your calculator, there is GRA. This is grades. A grade is another unit of angle equal to one hundredth of a right angle. Convert $3\pi/4$ radians to grades and 225 grades to radians.

Q52 Angular velocities are often measured in radians per second. A centrifuge is spinning at 200 radians per second. How long will it take to spin 1200 revolutions?

Q53 A light 35 km away is moving with an apparent speed of 0.0031 radians per second. Assuming that it is moving perpendicular to the line of sight, what is its speed? If it was known to be moving at 20° to the line of sight, what would its speed be then?

Revise

Revision Set 1

Q61 Convert the following to radians. Give the answers as decimals.

(a) 47° (b) 380° (c) 75°

Q62 Convert the following to radians. Give the answers as fractions of π.

(a) 60° (b) 315° (c) 75°

Q63 Convert the following to degrees.

(a) 2.1 (b) $3\pi/4$ (c) $14\pi/9$

Q64 Find the following with the calculator set in radians

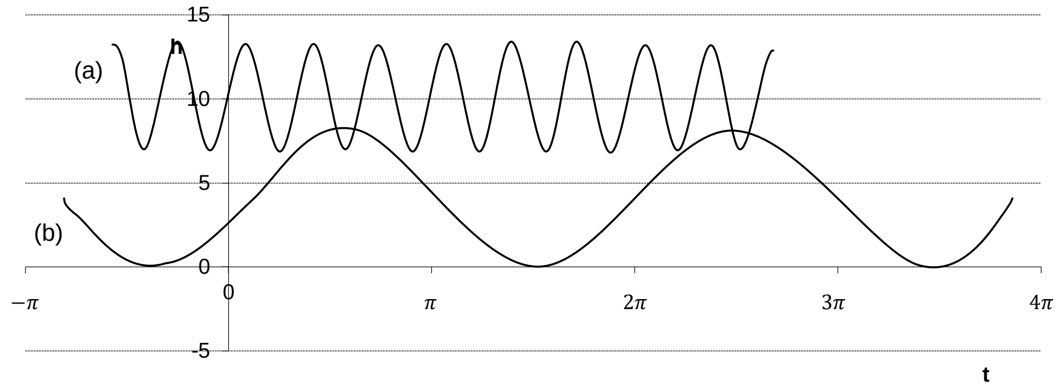
(a) tan 0.7 (b) $\cos 3\pi/8$ (c) $\sin^{-1} -0.5$

Q65 For each of the functions below,

(i) give the mean position, the amplitude, the period and the phase shift,

(ii) then find the values of the parameters a, b, c and d,

(iii) then find the formula.



Q66 Solve $3 \sin 2x = 1.5$ for $0 \le x \le 2\pi$.

Q67 Simplify: (a) $\cos(2\pi - \theta)$ (b) $\frac{\sin(\pi-\theta)}{\cos(\pi+\theta)}$

Answers

Q1	(a) 3.14	(b) 1.57	(c) 0.70	(d) 0.56
	(e) 1.75	(f) 0.87	(g) 3.93	(h) 0.12
	(i) 9.42	(j) 2.44	(k) 0.32	(l) 0.22
	(m) 1.26	(n) 2.15	(o) 0.63	(p) 8.90
Q2	(a) π	(b) $\pi/2$	(c) $\pi/3$	(d) $-\pi/6$
	(e) $2\pi/3$	(f) $\pi/4$	(g) $3\pi/2$	(h) 2π
	(i) 3π	(j) $5\pi/6$	(k) $-\pi/10$	(l) $\pi/15$
	(m) $2\pi/5$	(n) $123\pi/180$	(o) $\pi/5$	(p) $-11\pi/6$
Q3	(a) 180°	(b) 360°	(c) 45°	(d) 135°
	(e) 30°	(f) 20°	(g) 150°	(h) 1080°
	(i) 900°	(j) 240°	(k) 585°	(l) 1980°
	(m) 75°	(n) 270°	(o) –90°	(p) –135°
	(q) 155°	(r) 115°	(s) 266°	(t) –693°
Q4	(a) 0.565	(b) 0.454	(c) 0.932	(d) 0
	(e) 1	(f) 0.5	(g) 0.727	(h) 0.383
	(i) 0.707	(j) nd	(k) 0.524	(l) 1.05
	(m) 0.785	(n) 0.927	(o) 1.87	(p) −0.524
Q5	(a) (i) 11, 4, $\pi/4$, 0	(ii) 4, 8, 0, 11	(iii) $h = 4\cos 8t + 11$	
	(b) (i) 7, 3, π, 0	(ii) 3, 2, 0, 7	(iii) $h = 3\sin 2t + 7$	
	(c) (i) 1, 6, π, $\pi/8$	(ii) 6, 2, $-\pi/8$, 1	(iii) $h = -6\cos 2(t - \pi/8) + 1$	
	(d) (i) −4, 2, 2π, $-\pi/8$	(ii) 2, 1, $\pi/8$, – 4	(iii) $h = 2\cos(t + \pi/8) - 4$	
Q6	(a) (i) 12.5, 2.5, π, –0.3	(ii) 2.5, 2, 0.3, 12.5	(iii) $h = -2.5\sin 2(t + 0.3) + 12.5$	

(b) (i) 5, 3, $\pi/2$, − 0.2 (ii) 3, 4, 0.2, 5 (iii) $h = 3\cos(t + 0.2) + 5$

(c) (i) 3, 2, 8π, 0 (ii) 2, ¼, 0, 3 (iii) $h = -2\cos\frac{1}{4}t + 3$

(d) (i) −3, 3, 2π, 0 (ii) 3, 1, 0, −3 (iii) $h = 3\sin t - 3$

Q8 (a) $\pi/6$ or $5\pi/6$ (b) 1.50 or 4.79
(c) 1.94 or 5.91 (d) 0.83 or 3.36
(e) $\pi/6$ or $\pi/3$ or $7\pi/6$ or $4\pi/3$ (f) $3\pi/4$ or $7\pi/4$
(g) $5\pi/6$ or $11\pi/6$ (h) 0.38 or 3.52

Q9 (a) $\sin\theta$ (b) $-\cos\theta$ (c) $\sin\theta$ (d) $-\tan\theta$
(e) $\cos\theta$ (f) $\tan\theta$ (g) $-\sin\theta$ (h) $-\cos\theta$

Q10 (a) $\cos\theta$ (b) $-\sin\theta$ (c) $-\cos\theta$ (d) $\sin\theta$
(e) $-\cos\theta$ (f) $\sin\theta$ (g) 1 (h) $\tan\theta$
(i) $\cos\theta$ (j) $-\cos\theta$

Q51 150 grades, $9\pi/8$ radians
Q52 37.7 s
Q53 109 m/s, 317 m/s

Q61 (a) 0.82 (b) 6.63 (c) 1.31
Q62 (a) $\pi/3$ (b) $7\pi/4$ (c) $5\pi/12$
Q63 (a) 120° (b) 135° (c) 280°
Q64 (a) 0.84 (b) 0.38 (c) −0.52
Q65 (a) (i) 10, 3, $\pi/3$, 0 (ii) 3, 6, 0, 10 (iii) $h = 3\sin 6t + 10$
(b) (i) 4, 4, 2π, 0.5 (ii) 4, 1, -0.5, 4 (iii) $h = 4\sin(t - 0.5) + 4$
Q66 $x = \pi/12, 5\pi/12, 13\pi/12, 17\pi/12$
Q67 (a) $\cos\theta$ (b) $-\tan\theta$