M1 Maths

M5-2 Solving Triangles

- cosine and sine rules
- area formulae

Summary Learn Solve Revise Answers

Summary

To solve a triangle (i.e. to find any or all of the sides and angles), three quantities (sides and/or angles) have to be given. The three given quantities and the quantity needed are the involved quantities.

If the involved quantities are 3 sides and 1 angle, use the **cosine rule**:
$o^2 = a^2 + b^2 - 2ab \cos \theta$. Substitute into the formula, then solve.

If the involved quantities are 2 sides and 2 angles, use the **sine rule**.

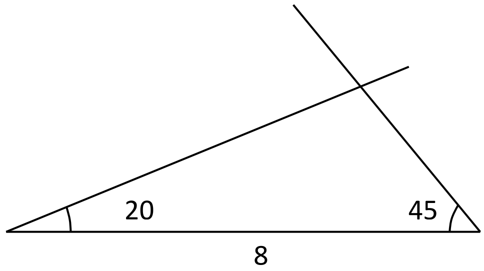
- to find a side, write in the third angle (using sum = 180°), then sub into $\frac{a}{\sin A} = \frac{b}{\sin B}$ and solve.
- to find an angle, sub into $\frac{\sin A}{a} = \frac{\sin B}{b}$ and solve to find one of the unknown angles (the other can be found using sum = 180°). Check with a sketch whether there are one or two possible solutions.
  If there are not two suitable side-angle pairs, use the cosine rule to find the third side, then use the sine rule, checking with a sketch to see if the required angle is acute or obtuse.

Apart from $A = b \times h \div 2$, there are two other formulae for the **area of a triangle**:
$A = ½ab \sin C$, where C is the angle between sides a and b, and
$A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the side lengths and s is the semi-perimeter.

Learn

Rules for solving general triangles

A general triangle (i.e. one that isn't necessarily right-angled) has six quantities – three side lengths and three angles. If we fix any three of these quantities, then the shape and size of the triangle is fixed and we can calculate any of the other quantities. See the example to the right.



[Note that fixing the three angles isn't adequate, because we can really only fix two. Once we've done that, there is no longer any choice for the third one anyway.]

So, it is possible to calculate any quantity in a triangle if we know three others. We call the three quantities we know and the quantity we wish to calculate 'the involved quantities'. In any triangle calculation, there will be four involved quantities.

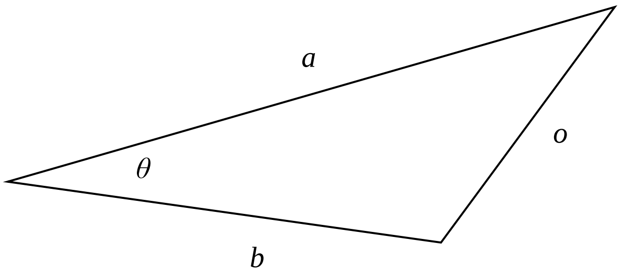
There are two rules for solving general triangles.

We use the **cosine rule** if the four involved quantities are three sides and one angle.

We use the **sine rule** if the four involved quantities are two sides and two angles.

Cosine Rule

We use the cosine rule when the involved quantities are 3 sides and one angle. Let's call the involved angle θ (theta). Let's call the side opposite the involved angle o and let's call the sides adjacent to the involved angle a and b. Like this:



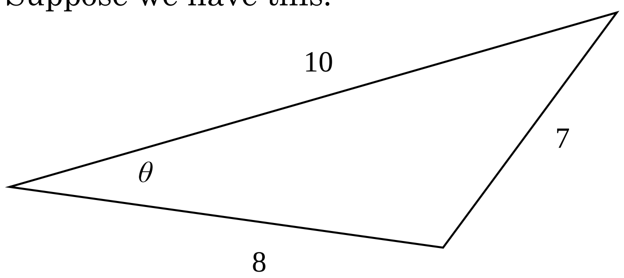
The cosine rule is then written $o^2 = a^2 + b^2 - 2ab \cos \theta$.

We just substitute the three known quantities into the formula and solve the resulting equation to get the unknown quantity.

There are three possible situations: we might need to find the angle, or we might need to find the opposite side, or we might need to find an adjacent side. We will look at these three in turn.

Using the cosine rule to find an angle

Suppose we have this:



Our working will look like this:

$o^2 = a^2 + b^2 - 2ab \cos \theta$

$7^2 = 10^2 + 8^2 - 2 \times 10 \times 8 \times \cos \theta.$

$49 = 100 + 64 - 160 \cos \theta.$

$-115 = -160 \cos \theta.$

$0.719 = \cos \theta.$

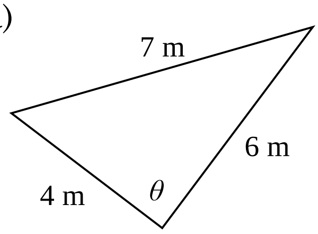
$44° = \theta.$

Be careful with the order of operations. A common mistake is to simplify 100 + 64 – 160 to 4 after line 3. But this is adding and subtracting before multiplying.

Practice

Q1 Find the angle marked θ in each of these triangles.

(a)



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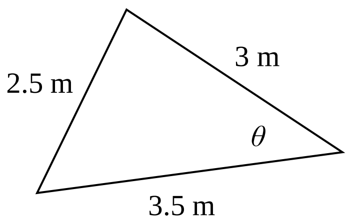
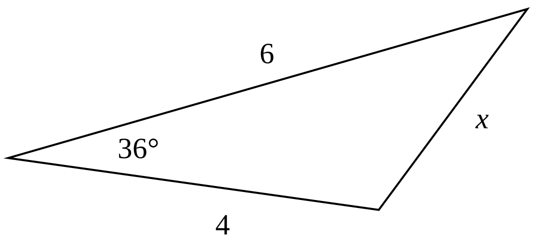


Using the cosine rule to find the side opposite the known angle

Suppose we have this:



Our working will look like this:

$o^2 = a^2 + b^2 - 2ab \cos \theta$

$x^2 = 6^2 + 4^2 - 2 \times 6 \times 4 \times \cos 36°.$

$x^2 = 36 + 16 - 48 \cos 36°.$

$x^2 = 52 - 48 \cos 36°.$

$x^2 = 13.167$

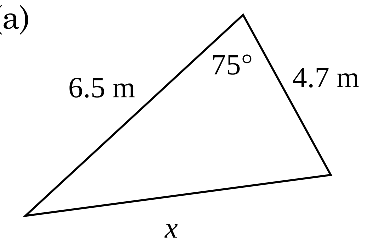
$x = 3.63$

Note that we don't need to worry about the negative solution to $x^2 = 13.167$ because sides of triangles cannot have negative lengths.

Practice

Q2 Find the side marked x in each of these triangles.

(a)



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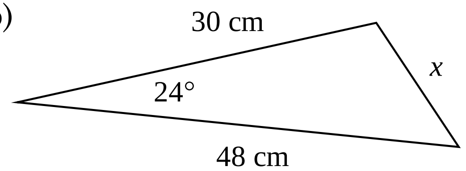
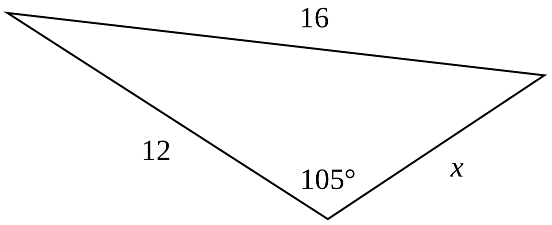


Using the cosine rule to find a side adjacent to the known angle

Suppose we have this:



Our working will look like this:

$$o^2 = a^2 + b^2 - 2ab\cos\theta$$

$$16^2 = 12^2 + x^2 - 2 \times 12 \times x \times \cos 105°.$$

$$256 = 144 + x^2 - 24 \times \cos 105° \times x$$

$$256 = 144 + x^2 - 24 \times \cos 105° \times x$$

$$256 = 144 + x^2 + 6.212\,x$$

$$x^2 + 6.212\,x - 112 = 0$$

This is a quadratic equation. In general, these equations will not be easily factorisable, so we use the quadratic formula.

$$a = 1, \quad b = 6.212, \quad c = -112$$

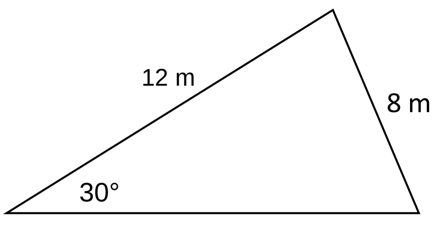
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

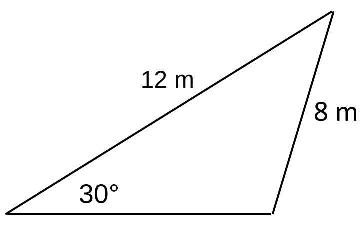
$$x = \frac{-6.212 \pm \sqrt{6.212^2 + 448}}{2}$$

$$x = 7.92 \quad or \quad x = -14.14$$

As x cannot be negative, we ignore the negative solution and conclude that $x = 7.92$.

Note, however, that sometimes you might get two positive solutions. In this case, there will be two possible values for x. The two triangles below are both possible with the given information.

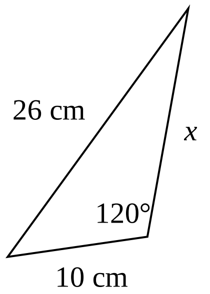




Practice

Q3 Find the side marked x in each of these triangles.

(a)



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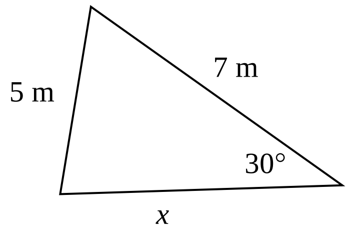
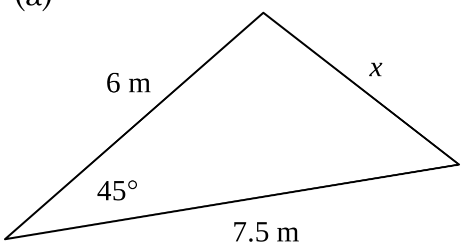


Here are some mixed cosine rule problems.

Practice

Q4 Find the unknown in the following triangles.

(a)



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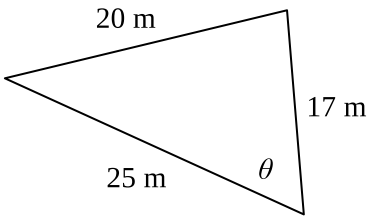

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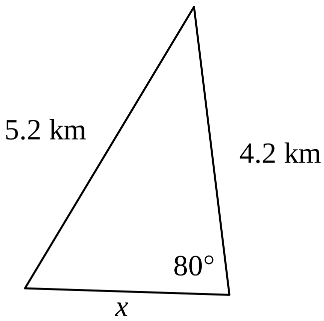

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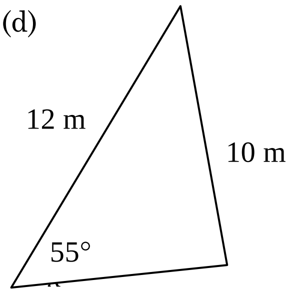


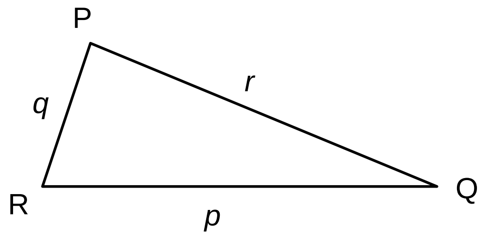
Sine Rule

Remember we use the cosine rule when the involved quantities are 3 sides and one angle. We use the sine rule when the involved quantities are two sides and two angles.

The sine rule states that the ratio of the sine of an angle to the length of the side opposite the angle is the same for all side-angle pairs in a triangle.

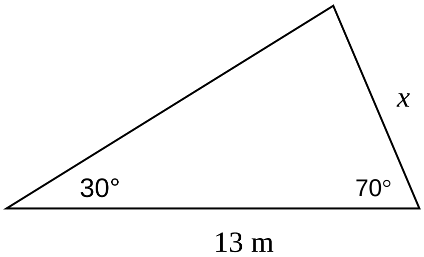
$$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ or } \frac{\sin A}{a} = \frac{\sin B}{b}$$ where side a is opposite angle A etc.

Any other letters can be substituted for a and A etc. as long as the fraction contains a side and its opposite angle. For instance, here we could use $\frac{p}{\sin P} = \frac{r}{\sin R}$

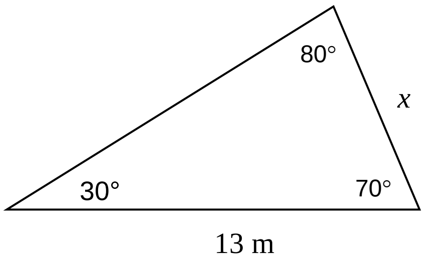


Using the sine rule to find a side

Suppose we have this:



Write in the third angle of the triangle (using the fact that the sum of the angles is 180°).



Then we can see that we have a known side-angle pair (13 m, 80°) and a half-known side-angle air (x, 30°).

Sometimes you will have this to begin with (e.g. if we had been given the left angle and the top angle). In those cases, you don't need to write in the third angle, though it can be quicker to do so anyway than to decide whether it's necessary.

Now we can write $\frac{a}{\sin A} = \frac{b}{\sin B}$ and substitute in the values:

$$\frac{x}{\sin 30} = \frac{13}{\sin 80}$$

Then we solve the equation:

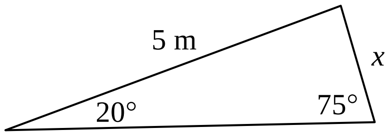
$$x = \frac{13 \sin 30}{\sin 80}$$

$$x = 6.6$$

Practice

Q5 Find the side marked x in these triangles.

(a)



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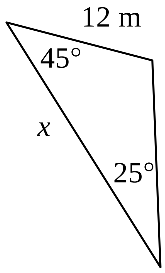
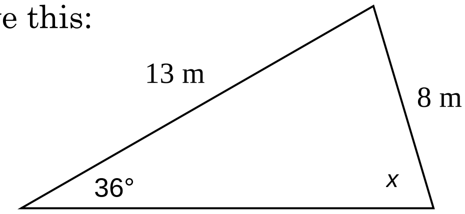


Using the sine rule to find an angle

Using the sine rule to find an angle can be more complicated and you have to keep your wits about you.

Case 1: Suppose we have this:



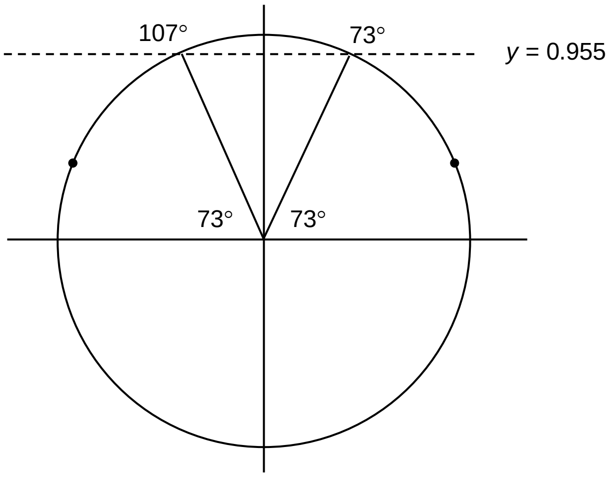
Here we have a known side-angle pair (8 m, 36°); and a half-known side-angle pair (13 m, x°). So we can use the sine rule to find the angle x.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin x}{13} = \frac{\sin 36}{8}$$

$$\sin x = \frac{13 \sin 36}{8}$$

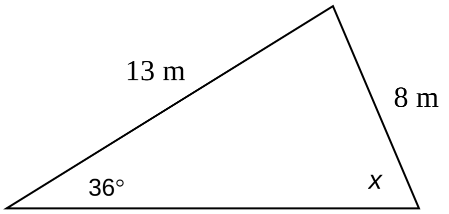
$$\sin x = 0.955$$

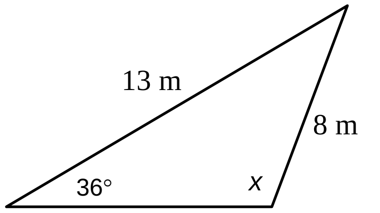


Now the calculator will give us $x = 73°$, but we know that $\sin x = 0.955$ has two solutions between 0 and 180°: $x = 73°$ and $x = 107°$.

We have to decide whether the angle is 73° or 107°.

Looking at the information on the diagram, we can see that both are possible. The triangle could look like either of those below.

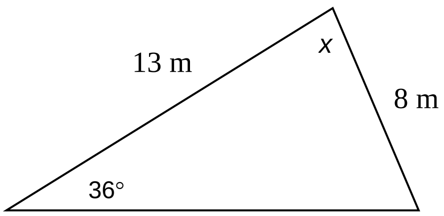




So there are two possible values for x. They are 73° and 107°.

In fact, there will often be two solutions when using the sine rule to find an angle.

Case 2: Now look at this triangle.



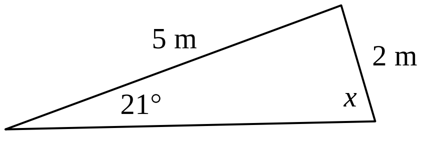
Here we have a known side-angle pair (8 m, 36°), but we don't have a half-known one. So we cannot find x directly with the sine rule. We have to find the other unknown angle first. We can call this third angle y, then use the sine rule to find that $y = 73°$ or 107°.

Then $x = 180 - 36 - 73 = 71°$ or $x = 180 - 36 - 107 = 37°$

Practice

Q6 Find the angle marked x in the following triangles, giving both solutions if there are two.

(a)



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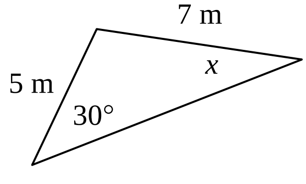

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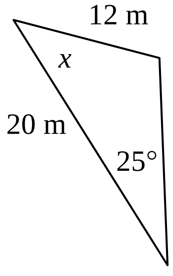

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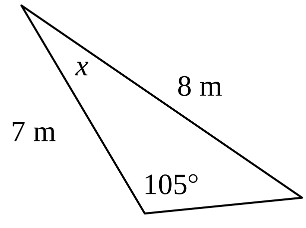
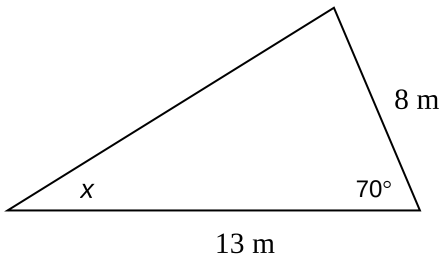


Case 3: Now look at this triangle.



The involved quantities are two sides and two angles, so we should use the sine rule. But we don't have a known side-angle pair, so we can't. What's more, if we tried to find the other angle first, we wouldn't have a known side-angle pair then either.

What we have to do here is find the length of the third side using the cosine rule. Then we will have a known side-angle pair and a half-known side-angle pair. Then we can use the sine rule (or the cosine rule if we wish) to find the angle x.

We would proceed like this:

Let the length of the unmarked side be s.

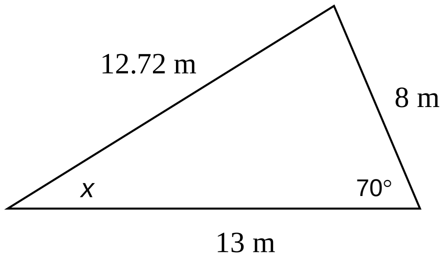
$$o^2 = a^2 + b^2 - 2ab \cos \theta$$

$$s^2 = 13^2 + 8^2 - 2 \times 13 \times 8 \times \cos 70°$$

$$s^2 = 169 + 64 - 208 \cos 70°$$

$$s^2 = 161.9$$

$$s = 12.72$$



Now we can use the sine rule.

As the unknown is an angle, it is best to use the sine rule in the form $\frac{\sin A}{a} = \frac{\sin B}{b}$

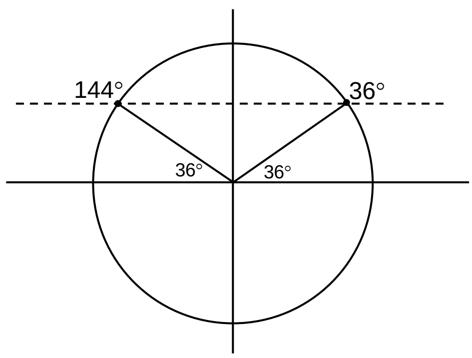
$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin x}{8} = \frac{\sin 70}{12.72}$$

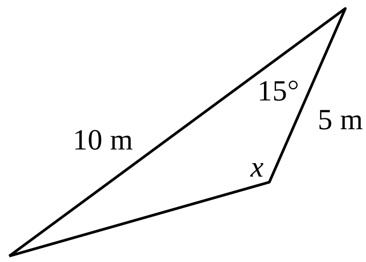
$$\sin x = \frac{8 \sin 70}{12.72}$$

$$\sin x = 0.591$$

Now we remember that there are always 2 angles less than 180° with a given sine. In this case $\sin^{-1} 0.591$ = 36° or 144°. We need to look at the diagram to see which is possible: it could be either of them. In this case, we can see it can't be 144°. So x = 36°.



But suppose we had this triangle:



Using the cosine rule, we would find that the bottom side is 5.3 m. Then, using the sine rule, we would have:

$$\frac{\sin x}{10} = \frac{\sin 15}{5.3}$$

$\sin x = 0.488$

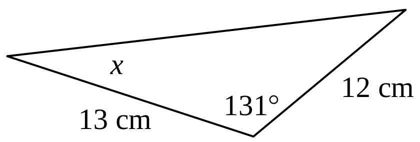
The calculator then gives us $x = 29°$.

But, from the diagram, it is obvious that x is obtuse, so we would take the 151° option rather than the 29° option for the angle.

Practice

Q7 Find the angle marked x in the following triangles.

(a)



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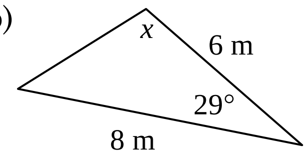


Summary

There are a number of different scenarios with the cosine and sine rules. They can be summarised as follows.

Involved quantities are 3 sides and 1 angle:
use the cosine rule: substitute then solve.

Involved quantities are 2 sides and 2 angles:
use the sine rule

- to find a side: write in the third angle, then sub and solve
- to find an angle where it's possible to write an equation using the sine rule (using either the required angle or finding the third angle first), do this, then solve it; use the sketch to see if the angle is acute, obtuse or could be either.
- to find an angle where it isn't possible to write an equation using the sine rule, use the cosine rule to get the third side, then use either the sine or cosine rule for the required angle.

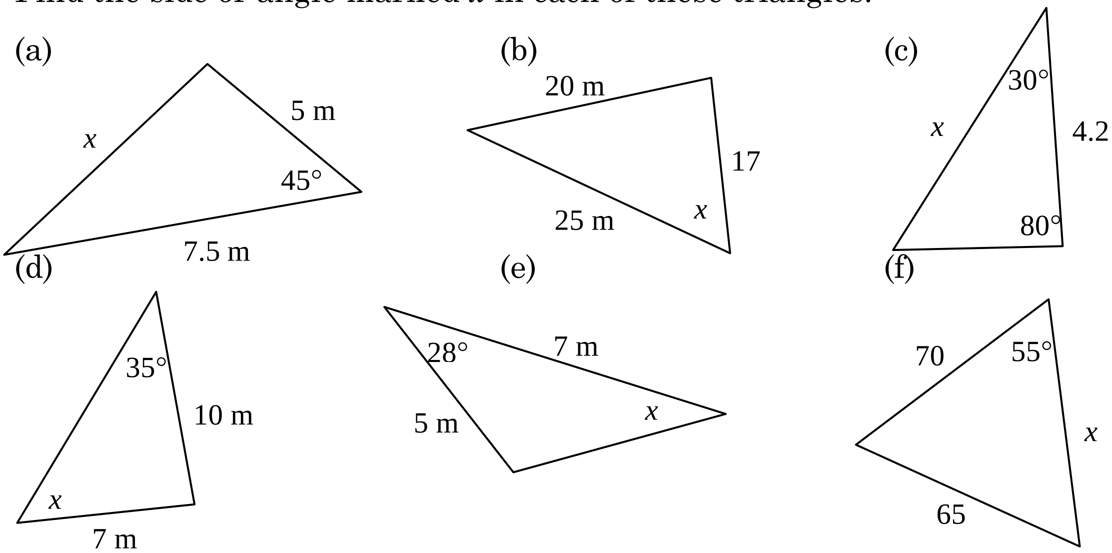
Now this probably seems complicated. But the good news is that you don't really have to remember it all. When faced with a sine or cosine rule problem, if you just try a few things, you will eventually get there.

Mixed Problems

Here are some mixed cosine and sine rule problems.

Practice

Q8 Find the side or angle marked x in each of these triangles.



Now we will apply what we have learnt to some real-life problems. In each case, the first job is to represent the situation with a diagram that contains a triangle with three known quantities marked and one unknown quantity marked. Then you apply the cosine and/or sine rule to the triangle.

Practice

Q9 A wire from the top of a tower to the ground to its south makes an angle of 30° with the ground. Another wire from the top of the tower to the ground to its north makes an angle of 50° with the ground. The points where the wires are attached to the ground are 42 m apart. Find the length of each wire and the height of the tower.

Q10 A robot is programmed to move 12 m in a straight line, then turn 140° to the right, then move another 7 m in a straight line.

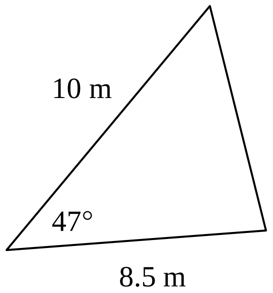
(a) How far would it then be from where it started?

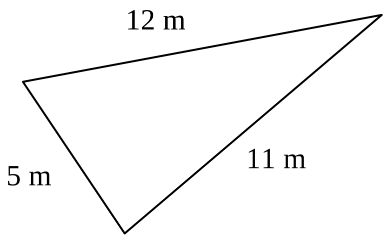
(b) How many degrees would it have to turn to the right to head towards where it started?

Q11 A ship sails 25 km on a bearing of 75°, then turns and sails on a bearing of 140° until it is due east of its starting point. How far from its starting point is it?

Area Formulae

Apart from $A = b \times h \div 2$, there are two other formulae for the area of a triangle:
$A = \frac{1}{2}ab \sin C$, where C is the angle between sides a and b, and
$A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the side lengths and s is the semi-perimeter ($(a + b + c) \div 2$).
The choice of which to use depends on what is known about the triangle.





For the triangle on the left,

$A = \frac{1}{2}ab \sin C$

$= \frac{1}{2} \times 10 \times 8.5 \times \sin 47°$

$= 31.1 \text{ m}^2$

For the triangle on the right,

$A = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{14(14-12)(14-11)(14-5)}$

$= \sqrt{14 \times 2 \times 3 \times 9}$

$= \sqrt{756}$

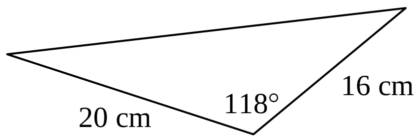
$= 27.5 \text{ m}^2$

The formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ is known as Heron's formula.

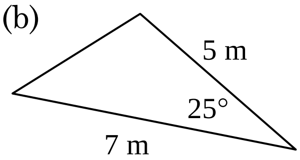
Practice

Q12 Find the areas of these triangles.

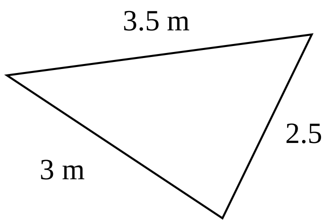
(a)



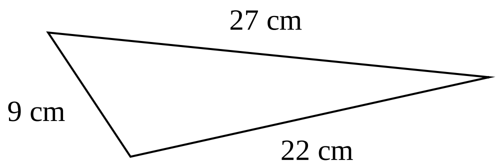
(b)



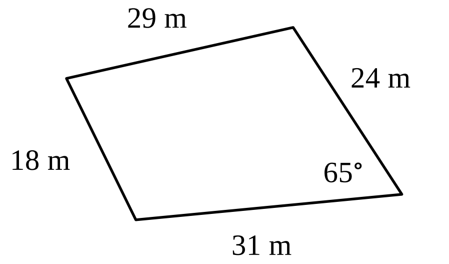
(c)



(d)



Q13 Find the area of this block of land.



Solve

Q51 A buoy is floating on a lake. It is attached to the bottom by two ropes, one 16 m long and one 24 m long. Both ropes are tight and are attached to the bottom 30 m apart. How deep is the lake?

Q52 Jo takes a bearing on a hill and finds it is at 340°. She then walks 3 km due west and finds the bearing of the same hill is now 25°. How much further west will she have to walk for the hill to be on a bearing of 45°?

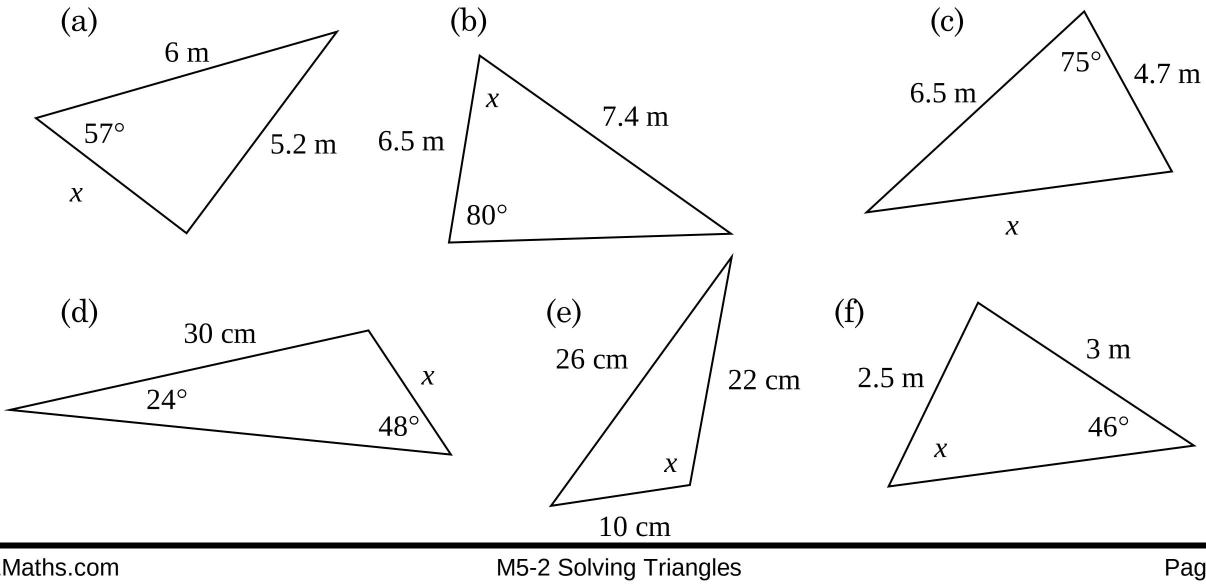
Q53 Sedgeworth is on a bearing of 245° from Hoddsville. Krilly and Mambo, two separate towns, are both on a bearing of 290° from Hoddsville. Sedgeworth is 220 km from Hoddsville and 185 km from both Krilly and Mambo. How far is Krilly from Mambo?

Q54 Prove the formula $A = \frac{1}{2}ab \sin C$.

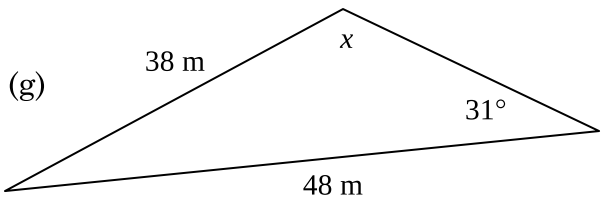
Revise

Revision Set 1

Q61 Find the side or angle marked x in each of the following triangles.



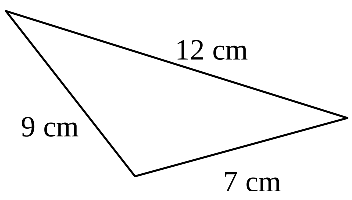
(g)



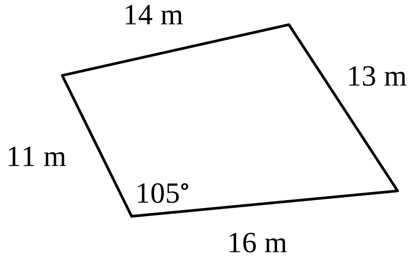
Q62 Hugo walked 2 km NE, then 4 km on a bearing of 190°. On what bearing should he walk to get back to where he started? How far would it be?

Q63 Find the areas of these shapes:

(a)



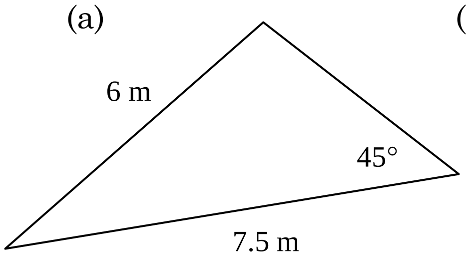
(b)



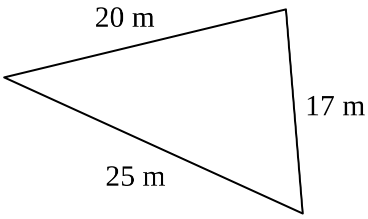
Revision Set 2

Q71 Solve the following triangles, i.e. find all the unknown sides and angles.

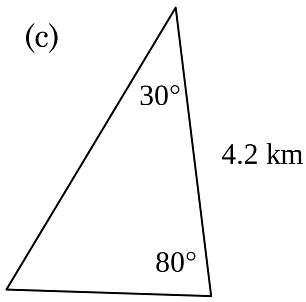
(a)



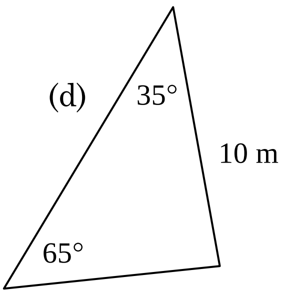
(b)



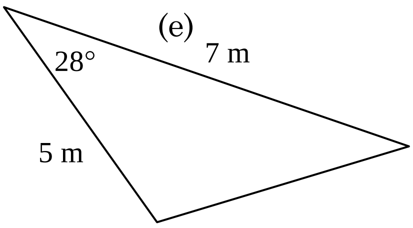
(c)



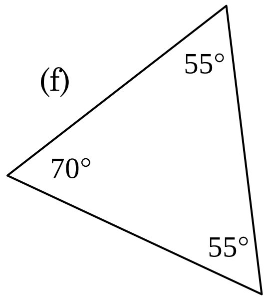
(d)



(e)



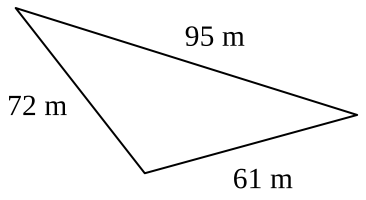
(f)



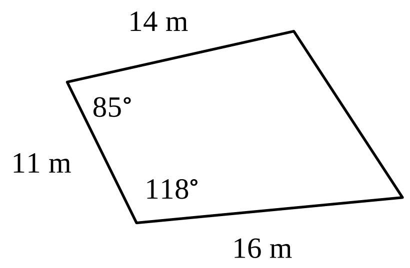
Q72 Janine stood on the bank of a river and saw a tree on the far bank on a bearing of 332°. She then walked 56 m along the bank following a bearing of 280° and found the same tree was now on a bearing of 017°. How wide is the river?

Q73 Find the areas of these shapes:

(a)



(b)



Answers

Q1	(a) 86°	(b) 44°		
Q2	(a) 7.0 m	(b) 24 cm		
Q3	(a) 19.5 cm	(b) 9.5 cm or 2.6 cm		
Q4	(a) 5.3 m	(b) 53°	(c) 2.4 km	(d) 8.7 m or 5.0 m
Q5	(a) 1.77 m	(b) 26.7 m		
Q6	(a) 64°, 116°	(b) 21°	(c) 20°, 110°	(d) 17°
Q7	(a) 23°	(b) 104°		
Q8	(a) 10.4 m	(b) 53	(c) 4.4 km	
	(d) 55°, 125°	(e) 42°	(f) 15 m or 44 m	
Q9	32.6 m, 21.3 m, 16.3 m			
Q10	(a) 8.0 m	(b) 106°		
Q11	29.3 km			
Q12	(a) 141 cm^2	(b) 7.4 cm^2	(c) 3.67 m^2	(d) 90.1 cm^2
Q13	590.13 m^2			
Q51	12.7 m			
Q52	1.9 km			
Q53	200 km			
Q61	(a) 2.0 m or 4.6 m	(b) 60°	(c) 7.0 m	
	(d) 16.4 cm	(e) 102°	(f) 60° or 120°	
	(g) 139°			
Q62	320°, 3.29 km			
Q63	(a) 31.3 cm^2	(b) 172.3 m^2		

Q71 From left to right:

(a) 73°, 62°, 8.11 m	(b) 43°, 85°, 53°	(c) 70°, 4.4 km, 2.2 km
(d) 10.9 m, 6.3 m, 80°	(e) 70°, 3.5 m, 82°	(f) not possible

Q72 62 m

Q73 (a) 2196 m^2 (b) 198 m^2