M1 Maths

M5-1 Unit Circle and Trig Identities

- sines, cosines and tangents in terms of the unit circle
- identities: tan ϑ = sin ϑ /cos ϑ, Pythagorean identity, sin ϑ = cos (90 – ϑ)

Summary Learn Solve Revise Answers

Summary

A unit circle is a circle with radius 1 centred on the origin, O of the Cartesian plane. If P is a point (x, y) on the circumference, then the angle θ is defined as the anticlockwise turn from the positive x-axis to OP.

sin θ can then be defined as y and cos θ can be defined as x. This provides a definition of the trig functions for all angles.

Circle diagrams are sketches of the unit circle with other relevant information. They can be used to estimate sines and cosines of any angle and to estimate the angles which have given sines and cosines. A calculator will give just one angle corresponding to a given sine or cosine. The circle diagram can be used to find others.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ and so is the gradient of the line OP. Circle diagrams can be used to estimate the tan of any angle and to estimate the angle for any tan as well as to find angles not given by the calculator.

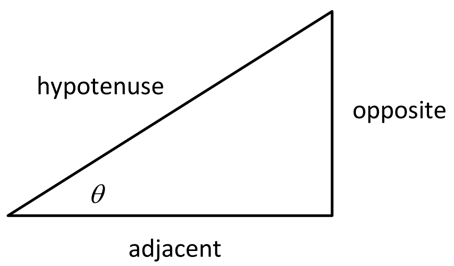
The Pythagorean identity is $\sin^2\theta + \cos^2\theta = 1$ and is true for all θ. It allows us to find the other trig values for an angle from the sine or cosine.

The sine of an angle is equal to the cosine of its complement and vice versa.

Learn

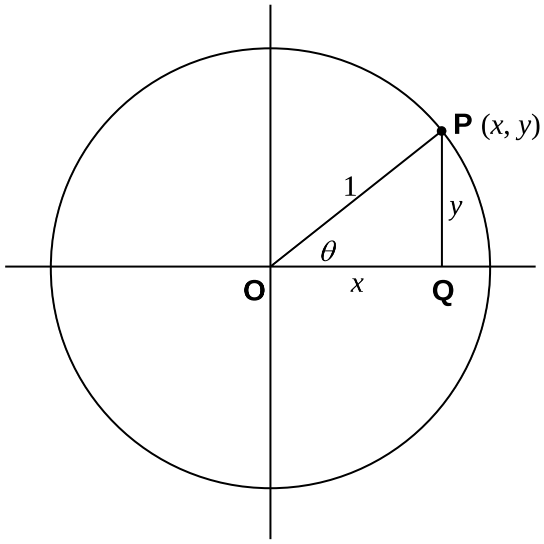
A new definition of sin θ

You already know that $\sin\theta = \frac{opposite}{hypotenuse}$



Consider a circle with radius 1 (a unit circle) centred on the origin of the Cartesian plane, O.

Consider a radius OP at an angle θ anticlockwise from the positive direction of the x-axis. Let the outer end of the radius be the point P at coordinates (x, y).



$\sin\theta = \frac{PQ}{OP} = \frac{y}{1} = y$

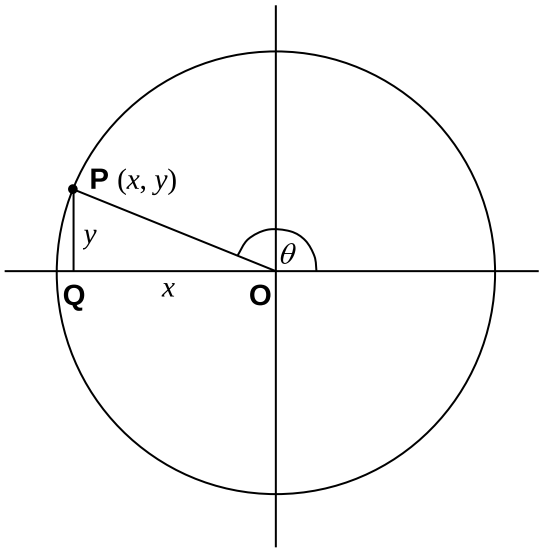
So $\sin\theta$ is the y-coordinate of P.

We originally defined $\sin\theta$ as $\frac{opposite}{hypotenuse}$.

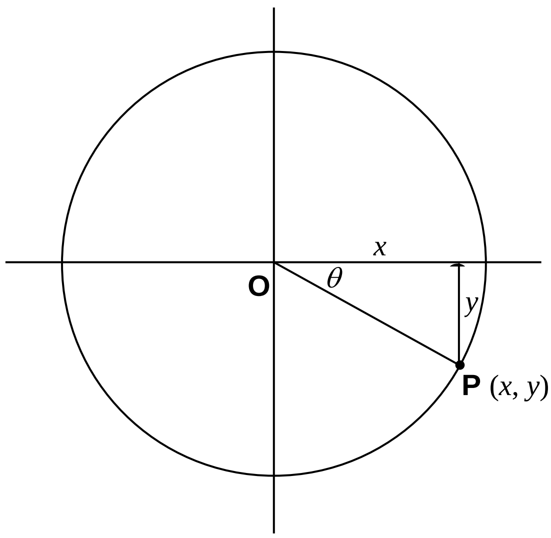
But we can also define $\sin\theta$ as the y-coordinate of P without contradicting the original definition.

The advantage of the new definition is that it can be applied to angles greater than 90°. In fact, it can be applied to angles of any size, even greater than 360° or less than 0°.

If $\theta = 160°$, then, as can be seen in the diagram to the right, $\sin\theta = y \approx 0.3$



If $\theta = -30°$, then, as can be seen in the next diagram, $\sin\theta = y \approx -0.5$



If $\theta = 400°$, then P moves 400° anticlockwise around the circle. This will be one and a bit revolutions. It will and up 40° around.

This means that $\sin 400° = \sin 40°$.

400° and 40° are known as co-terminal angles. This is because P is in the same place in both cases.

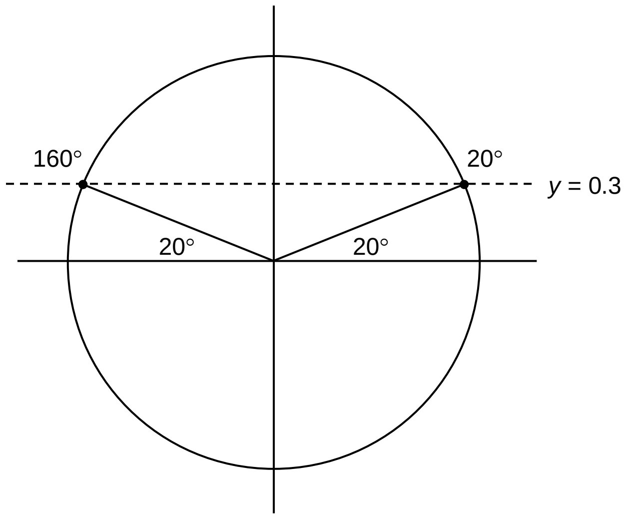
Practice

Q1 Sketch circle diagrams like the ones above to approximate the following sines. After each, use your calculator to find the sine and see how close you were.

(a) sin 30°	(b) sin 80°	(c) sin 130°	(d) sin 270°
(e) sin –40°	(f) sin –90°	(g) sin –120°	(h) sin 350°
(i) sin 400°	(j) sin 460°	(k) sin 780°	(l) sin –400°
(m) sin 70°	(n) sin 110°	(o) sin 200°	(p) sin 340°

Finding angles from their sines

What angle has a sine of 0.3?

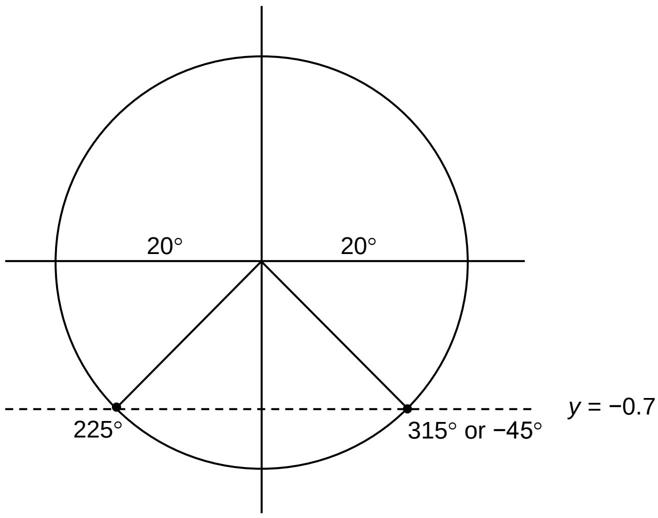


We can use the unit circle to get a rough answer to this question too.

We mark about 0.3 on the y-axis, then draw a horizontal line through it. We then mark the points where the horizontal line cuts the circle. These are our points P.

We then draw radii from the origin (centre of the circle) to each of these points P and estimate the angles for each of these points. In the diagram, we can see that the first is about 20°. Because the circle is symmetrical, this means that the second one is about 20° back from 180°, in other words about 160°.

If we have a negative sine, e.g. $\sin\theta = -0.7$, then our horizontal line will be below the x-axis and our angles will be in the lower quadrants.



Practice

Q2 Use circle diagrams to find the approximate angles between 0° and 360° inclusive which have the following sines.

(a) $\sin\theta = 0.6$ (b) $\sin\theta = 0.1$ (c) $\sin\theta = 0.92$

(d) $\sin\theta = 0.5$ (e) $\sin\theta = -0.5$ (f) $\sin\theta = -0.15$

(g) $\sin\theta = 0.97$ (h) $\sin\theta = 0$ (i) $\sin\theta = 0.44$

(j) $\sin\theta = -0.62$ (k) $\sin\theta = 1$ (l) $\sin\theta = -1$

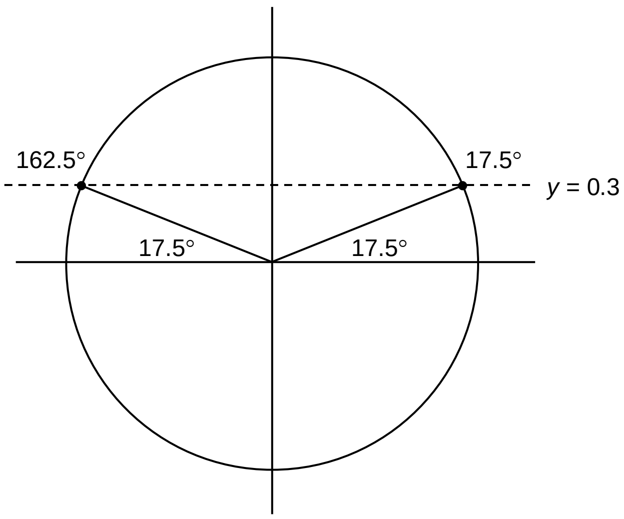
Of course, there are more angles than 20° and 160° that have a sine of 0.3. 380° does too because it is in the same place as 20° (co-terminal with 20°), just one revolution further round. Similarly 520° does, as do −200° and −340° and so on. In fact we can add or subtract 360° to/from any solution and get another solution. There are an infinite number of angles with a sine of 0.3.

In practice, we are mostly interested just in those between 0° and 360° inclusive (or sometimes between −180° and 180°), so we don't usually need to worry about all those others.

Using the Circle Diagram to Get Accurate Answers

Suppose we need to get accurate solutions to $\sin\theta = 0.3$. We can find out by keying in $\sin^{-1} 0.3$ on our calculator. It will give us an answer of about 17.5°.

But it won't give us the second angle (or any others). To get the second angle, we have to use a circle diagram. The symmetry of the circle tell us that if the first angle is 17.5° on from the positive x-axis, then the second angle is 17.5° back from the negative x-axis, i.e. 162.5°.



Practice

Q3 Use your calculator and a circle diagram to find the accurate angles (to the nearest degree) between 0° and 360° inclusive which have the following sines.

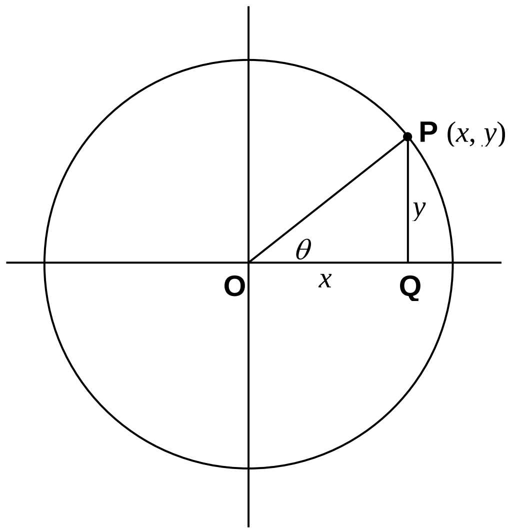
(a) $\sin\theta = 0.6$	(b) $\sin\theta = 0.1$	(c) $\sin\theta = 0.92$
(d) $\sin\theta = 0.5$	(e) $\sin\theta = -0.5$	(f) $\sin\theta = -0.15$
(g) $\sin\theta = 0.97$	(h) $\sin\theta = 0$	(i) $\sin\theta = 0.44$
(j) $\sin\theta = -0.62$	(k) $\sin\theta = 1$	(l) $\sin\theta = -1$

A new definition of cos θ

You already know that $\cos\theta = \frac{adjacent}{hypotenuse}$

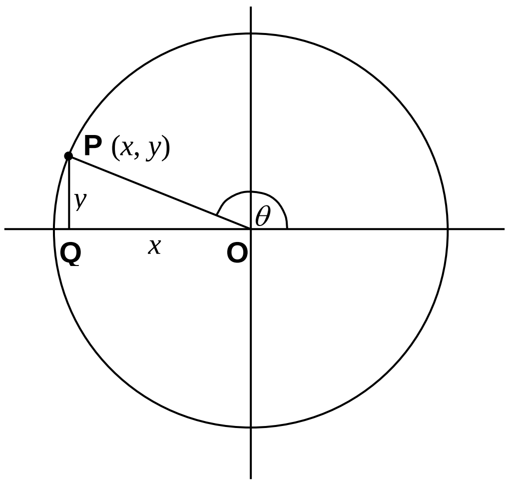
On the circle diagram, $\cos\theta = \frac{OQ}{OP} = \frac{x}{1} = x$

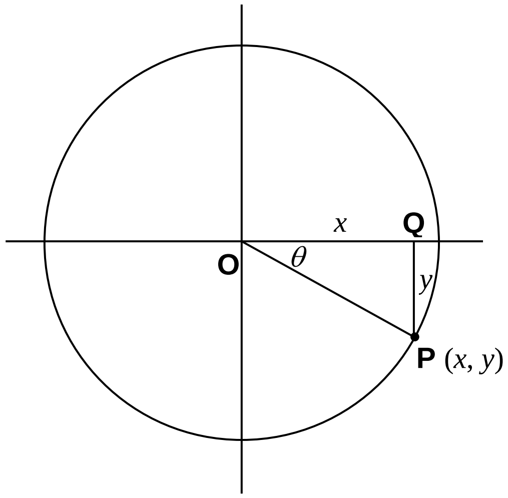
So $\cos\theta$ is the x-coordinate of P.



This is our new definition of $\cos\theta$ corresponding the new definition of $\sin\theta$.

If $\theta = 160°$, then, as can be seen in the diagram below left, $\cos\theta = x \approx -0.9$





If $\theta = -30°$, then, as can be seen in the diagram above right, $\cos\theta = x \approx 0.8$

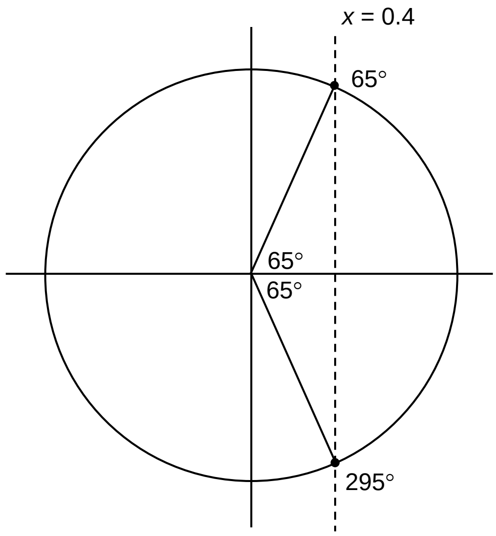
Practice

Q4 Draw circle diagrams like the ones above to approximate the following cosines. After each, use your calculator to find the cosine and see how close you were.

(a) cos 30°	(b) cos 80°	(c) cos 130°	(d) cos 270°
(e) cos –40°	(f) cos –90°	(g) cos –120°	(h) cos 350°
(i) cos 400°	(j) cos 460°	(k) cos 780°	(l) cos –400°
(m) cos 70°	(n) cos 110°	(o) cos 200°	(p) cos 340°

Finding angles from their cosines

What angle has a cosine of 0.4?



As with sine, we can use the unit circle to get a rough answer to this question.

We mark about 0.4 on the x-axis, then draw a vertical line through it. We then mark the points where the vertical line cuts the circle. These are our points P.

We then draw radii from the origin (centre of the circle) to each of these points P and estimate the angles for each of these points. In the diagram, we can see that the first is about 65°. Because the circle is symmetrical, this means that the second one is about 65° down from the x-axis, in other words about −65° or about 295°.

If we have a negative cosine, e.g. $\cos\theta = -0.8$, then our vertical line will be to the left of the y-axis and our angles will be in the left-hand quadrants.

Practice

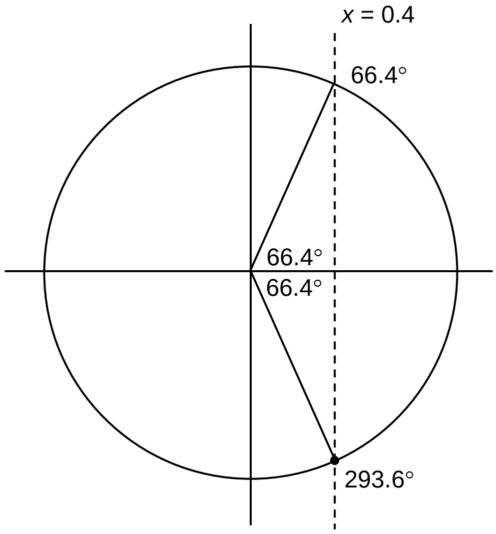
Q5 Use circle diagrams to find approximate angles between 0° and 360° inclusive which have the following cosines.

(a) $\cos\theta = 0.6$	(b) $\cos\theta = 0.1$	(c) $\cos\theta = 0.92$
(d) $\cos\theta = 0.5$	(e) $\cos\theta = -0.5$	(f) $\cos\theta = -0.15$
(g) $\cos\theta = 0.97$	(h) $\cos\theta = 0$	(i) $\cos\theta = 0.44$
(j) $\cos\theta = -0.62$	(k) $\cos\theta = 1$	(l) $\cos\theta = -1$

Using the Circle Diagram to Get Accurate Answers

Suppose we need to get accurate solutions to $\cos\theta = 0.4$ between 0° and 360°. We can do this by keying in $\cos^{-1} 0.4$ on our calculator. It will give us an answer of about 66.4°.

But again, it won't give us the second angle (or any others). Again, we have to use a circle diagram. The symmetry of the circle tells us that, if the first angle is 66.4° above the positive x-axis, then the second angle is 66.4° below the positive axis, i.e. 293.6°.



Practice

Q6 Use your calculator and a circle diagram to find the accurate angles between 0° and 360° inclusive which have the following cosines.

(a) $\cos\theta = 0.6$ (b) $\cos\theta = 0.1$ (c) $\cos\theta = 0.92$

(d) $\cos\theta = 0.5$ (e) $\cos\theta = -0.5$ (f) $\cos\theta = -0.15$

(g) $\cos\theta = 0.97$ (h) $\cos\theta = 0$ (i) $\cos\theta = 0.44$

(j) $\cos\theta = -0.62$ (k) $\cos\theta = 1$ (l) $\cos\theta = -1$

tan θ

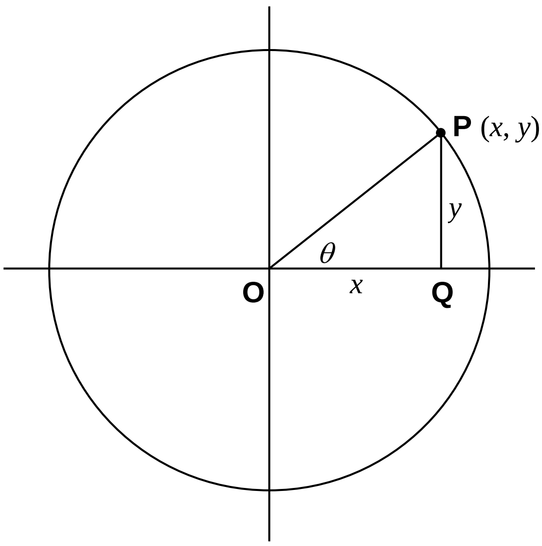
We tend to use circle diagrams less for $\tan\theta$ than for $\sin\theta$ or $\cos\theta$, but you still need to know how to do it.

$\tan\theta$ on the circle diagram is represented by the gradient of the line OP.

Why? Well, $\sin\theta = y$ and $\cos\theta = x$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (see box below) $= \dfrac{y}{x}$

which is the gradient of OP ($\dfrac{rise}{run}$).



$$\sin\,\theta = \frac{opposite}{hypotenuse} \quad \cos\,\theta = \frac{adjacent}{hypotenuse}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{opposite}{hypotenuse} \div \frac{adjacent}{hypotenuse}$$

$$= \frac{opposite}{hypotenuse} \times \frac{hypotenuse}{adjacent}$$

$$= \frac{opposite}{adjacent}$$

$$= \tan\,\theta$$

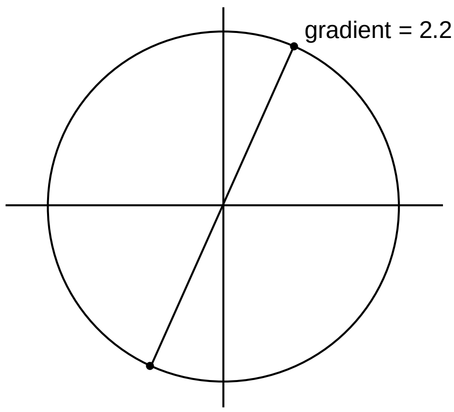
$\tan\,\theta = \frac{\sin\theta}{\cos\theta}$ is true for all values of θ and so is called an identity.

So, if we know θ, we can draw in the line OP, then just estimate its gradient.

[Remember that a horizontal line has a gradient of 0; a line at 45° to the horizontal has a gradient of 1; as lines approach vertical, their gradients approach infinity; lines that rise to the right have positive gradient and lines that fall to the right have negative gradient.]

If we know tan θ, we draw the line OP at that gradient, then just estimate the angle. In general there will be two angles between 0° and 360° which have a given tan. One will be 180° more than the other.

For instance, suppose we know tan θ = 2.2.
We draw the circle diagram like this:



The two angles will be roughly 70° and 70° + 180°, i.e. 250°.

If we need accurate values, we use the calculator to get the first one. $\tan^{-1} 2.2 = 66°$. The second angle is 66° + 180°, i.e. 246°.

If the calculator gives us a negative angle like –40°, we just add 360° to get 320°. The other angle will be 180° less, i.e. 140°.

Practice

Q7 Draw circle diagrams to approximate the following tangents. After each, use your calculator to find the tan and see how close you were.

(a) tan 30° (b) tan 80° (c) tan 130° (d) tan 270°

(e) tan –40° (f) tan –90° (g) tan –120° (h) tan 350°

Q8 Use circle diagrams to find approximate angles between 0° and 360° inclusive which have the following tangents.

(a) $\tan\theta = 0.6$ (b) $\tan\theta = -0.1$ (c) $\tan\theta = 2.4$

(d) $\tan\theta = -1$ (e) $\tan\theta = -10$ (f) $\tan\theta = 1.2$

Q9 Use your calculator and a circle diagram to find the accurate angles between 0° and 360° inclusive which have the following tangents.

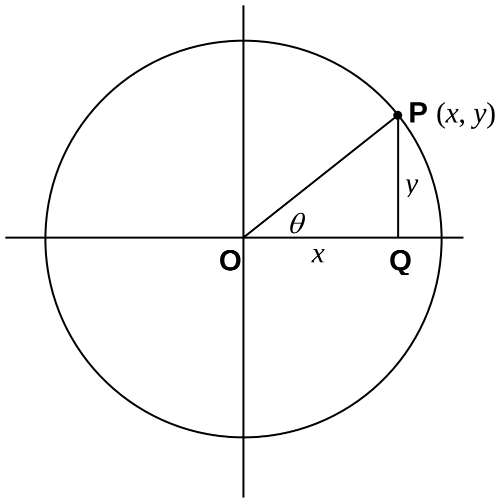
(a) $\tan\theta = 1.7$ (b) $\tan\theta = -0.2$ (c) $\tan\theta = 0.72$

(d) $\tan\theta = -3$ (e) $\tan\theta = 0.5$ (f) $\tan\theta = 0$

$\sin^2\theta + \cos^2\theta = 1$

Like $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\sin^2\theta + \cos^2\theta = 1$ is true for all values of θ, and so is also an identity.

[Note that $\sin^2\theta$ is a shorthand way of writing $(\sin\theta)^2$.]



This identity can be proved by using Pythagoras on the unit circle diagram.

By Pythagoras, $QP^2 + OQ^2 = OP^2$.

As the radius of the circle is 1, $OP^2 = 1$,
so $QP^2 + OQ^2 = 1$.

As QP = $\sin\theta$ and OQ = $\cos\theta$, $\sin^2\theta + \cos^2\theta = 1$.

This identity is often called the Pythagorean identity. It can also be written in the forms:

$\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$ and is most commonly used in these forms.

With the tan identity and the Pythagorean identity, it is possible, given the sin or cos of any angle, to find the other two trigonometric ratios.

For example, if $\sin \theta = 0.6$, then

$$\begin{aligned}\cos^2 \theta &= 1 - \sin^2 \theta \\ &= 1 - 0.6^2 \\ &= 1 - 0.36 \\ &= 0.64 \\ \cos \theta &= \pm 0.8\end{aligned}$$

$$\text{Then } \tan \theta = \frac{\sin\theta}{\cos\theta} = \pm \frac{0.6}{0.8} = \pm 0.75$$

Practice

Q10 Find the other two trigonometric ratios in each of the following cases. Don't use the sin, cos or tan buttons on your calculator.

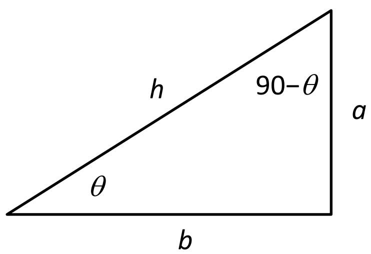
(a) $\sin \theta = 0.8$ (b) $\sin \theta = 0.1$ (c) $\sin \theta = 0.5$

(d) $\cos \theta = 0.5$ (e) $\cos \theta = 0.45$ (f) $\cos \theta = 0.22$

(g) $\sin \theta = -0.8$ (h) $\sin \theta = 0$ (i) $\sin \theta = -0.36$

(j) $\cos \theta = -0.62$ (k) $\cos \theta = 1$ (l) $\cos \theta = -1$

$\sin \theta = \cos (90 - \theta)$



In this right-angle triangle, $\sin \theta = {}^{a}/_{h}$ and $\cos (90 - \theta) = {}^{a}/_{h}$.

So $\sin \theta = \cos (90 - \theta)$. Also $\cos \theta = \sin (90 - \theta)$.

In other words the sine of an angle is the cosine of its complement and the cosine of an angle is the sine of its complement.

Practice

Q11 Use the complement identity to write the complementary trig value for an acute angle. E.g., given sin 30 = 0.5, write 'cos 60 = 0.5'.

(a) $\sin 20° = 0.34$ (b) $\sin 55° = 0.82$ (c) $\cos 25° = 0.91$

(d) $\cos 78° = 0.21$ (e) $\cos 90° = 0$ (f) $\sin 45° = 0.71$

Solve

Q51 Use the Pythagorean identity to prove that $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$.

Q52 Given that sin 30° = 0.5, write a single expression for all angles, θ, from $-\infty$ to ∞ that have a sine of 0.5. Hint: use n to represent any integer.

Q53 Solve $\sin\theta = 2\cos\theta$ for $0 \le \theta \le 360°$.

Revise

Revision Set 1

Q61 Use a unit circle to estimate the following. Use a calculator to check that you are close.

(a) sin 60° (b) cos 10° (c) tan 50° (d) sin 150°
(e) cos 260° (f) tan 110° (g) sin (−30°) (h) cos 290°

Q62 Use a unit circle to estimate the angles between 0 and 360 inclusive which have the following trig values:

(a) $\sin\theta = 0.8$ (b) $\sin\theta = -0.5$ (c) $\cos\theta = 0.1$ (d) $\cos\theta = -0.9$
(e) $\tan\theta = 1.2$ (f) $\tan\theta = -0.3$ (g) $\sin\theta = 0$ (h) $\cos\theta = 1$

Q63 Use a calculator and a unit circle to give all the angles between 0 and 360 inclusive with the following trig vales:

(a) $\sin\theta = 0.8$ (b) $\sin\theta = -0.5$ (c) $\cos\theta = 0.1$ (d) $\cos\theta = -0.9$
(e) $\tan\theta = 1.2$ (f) $\tan\theta = -0.3$ (g) $\sin\theta = 0$ (h) $\cos\theta = 1$

Q64 For each of the following, find all possible values of the other two trig functions:

(a) $\sin\theta = 0.8$ (b) $\sin\theta = -0.5$ (c) $\cos\theta = 0.1$ (d) $\cos\theta = -0.9$

Q65 If $\sin\theta = a$, find $\cos^{-1} a$.

Answers

Q2
(a) 37°, 143° (b) 6°, 174° (c) 67°, 113°
(d) 30°, 150° (e) 210°, 330° (f) 189°, 351°
(g) 76°, 104° (h) 0°, 180° (i) 26°, 154°
(j) 218°, 322° (k) 90° (l) 270°

Q3
(a) 37°, 143° (b) 6°, 174° (c) 67°, 113°
(d) 30°, 150° (e) 210°, 330° (f) 189°, 351°
(g) 76°, 104° (h) 0°, 180° (i) 26°, 154°
(j) 218°, 322° (k) 90° (l) 270°

Q5	(a) 53°, 307°	(b) 84°, 276°	(c) 23°, 337°
	(d) 60°, 300°	(e) 120°, 240°	(f) 98°, 262°
	(g) 14°, 346°	(h) 90°, 270°	(i) 64°, 296°
	(j) 128°, 232°	(k) 0°, 360°	(l) 180°
Q6	(a) 53°, 307°	(b) 84°, 276°	(c) 23°, 337°
	(d) 60°, 300°	(e) 120°, 240°	(f) 98°, 262°
	(g) 14°, 346°	(h) 90°, 270°	(i) 64°, 296°
	(j) 128°, 232°	(k) 0°, 360°	(l) 180°
Q8	(a) 31°, 211°	(b) 174°, 354°	(c) 67°, 247°
	(d) 135°, 315°	(e) 96°, 276°	(f) 50°, 130°
Q9	(a) 59°, 239°	(b) 169°, 349°	(c) 36°, 216°
	(d) 109°, 289°	(e) 27°, 207°	(f) 0°, 180°, 360°

Q10
(a) cos $\theta = \pm 0.60$, tan $\theta = \pm 1.33$
(b) cos $\theta = \pm 0.99$, tan $\theta = \pm 0.10$
(c) cos $\theta = \pm 0.87$, tan $\theta = \pm 0.58$
(d) sin $\theta = \pm 0.87$, tan $\theta = \pm 0.58$
(e) sin $\theta = \pm 0.89$, tan $\theta = \pm 1.98$
(f) sin $\theta = \pm 0.98$, tan $\theta = \pm 4.43$
(g) cos $\theta = \pm 0.60$, tan $\theta = \pm 1.33$
(h) cos $\theta = \pm 1$, tan $\theta = 0$
(i) cos $\theta = \pm 0.93$, tan $\theta = \pm 0.39$
(j) sin $\theta = \pm 0.78$, tan $\theta = \pm 1.27$
(k) sin $\theta = \pm 1$, tan θ not defined
(l) sin $\theta = 0$, tan $\theta = 0$

Q11	(a) cos 70° = 0.34	(b) cos 35° = 0.82	(c) sin 65° = 0.91
	(d) sin 12° = 0.21	(e) sin 0° = 0	(f) cos 45° = 0.71

Q52 $\theta = 90° \pm 60° + 2n°$, where n is any integer
Q53 $\theta = 63°$ or 243°

Q62	(a) 46°, 134°	(b) 210°, 330°	(c) 84°, 276°	(d) 154°, 206°
	(e) 50°, 230°	(f) 163°, 343°	(g) 0°, 180°, 360°	(h) 0°, 360°
Q63	(a) 46°, 134°	(b) 210°, 330°	(c) 84°, 276°	(d) 154°, 206°
	(e) 50°, 230°	(f) 163°, 343°	(g) 0°, 180°, 360°	(h) 0°, 360°

Q64
(a) cos $\theta = \pm 0.60$, tan $\theta = \pm 1.33$
(b) cos $\theta = \pm 0.87$, tan $\theta = \pm 0.58$
(c) sin $\theta = \pm 0.99$, tan $\theta = \pm 9.95$
(d) sin $\theta = \pm 0.44$, tan $\theta = \pm 0.48$

Q65 $90 - \theta$