M1Maths

M1Maths.com – Explanations and Practice for all school maths topics

M4-2 Errors

- no measurement is exact
- accuracy and precision
- random and systematic errors
- absolute and relative errors
- combining errors
- rounding during working

Summary Learn Solve Revise Answers

Summary

The number of units in any measurable quantity would have more non-zero decimal places than any measuring device could determine. Thus all measurements have an error.

Accuracy is how small the error is in a measurement. Precision is how closely further measurements of the same thing will agree with the original measurement.

If a measurement is repeated, random errors cause the measurements to be scattered either side of the mean of the measurements. A systematic error will cause the mean to be significantly different from the actual true value.

Absolute error is the difference between the measurement and the actual true value, given in measurement units. Relative error is the absolute error divided by the actual true value, given as a fraction or a percentage.

Measurements can be expressed with their uncertainty as say 14.2 cm ± 0.2 cm, where one is reasonably confident that the actual true value lies within 0.2 cm of 14.2 cm, i.e. between 14.0 and 14.4 cm. 14.0 and 14.4 cm are called the upper and lower bounds.

When adding or subtracting quantities with uncertainties, the absolute uncertainties are added to get the uncertainty of the result. When multiplying or dividing measurements with uncertainties, the relative uncertainties are added to get the relative uncertainty of the result.

When performing a calculation involving several numbers, one should not round any numbers before the final answer. The final answer should be rounded so as not to have more significant figures than is justified by the input data.

Learn

Every measurement has an error

The largest diamond ever found was the Cullinan diamond, weighing 621.20 g.

We might think this means that its mass to 8 decimal places was 621.20000000 g. But it probably wouldn't be. If we had a super-good measuring device, we might find that it weighs 621.20285107 g. The actual mass would have non-zero decimal places that go on a lot further than that and possibly for ever. No measuring device would be able to measure accurate to the last one.

This means that any measurement is just an approximation. There will always be an error. If the mass of the diamond was 621.20285107… g and we measured it at 621.20 g, then the error would be –0.00285107… g. We use a negative sign to show that the measurement is less than the actual mass.

error = *measured mass – actual mass*

The same goes for all other measurements too: length, volume, speed, temperature, time, electric current etc. No measurement can ever be exact: there will always be an error. By '**exact**', we mean that there is no error at all, not even the tiniest.

Note that a count can be exact. For instance, if there are 15 apples and we count 15, then that is exact: there is no error. This is because the number of apples is a whole number with no non-zero decimal places. But a measurement can't be exact – it will always have an error.

Practice

Q1 (a) If you count your fingers, is it possible to get an exact result?

(b) If you weigh a wedding ring, is it possible to get an exact result? Explain why.

(c) Is it possible to drive for 5 minutes at exactly 60 km/h? Explain why.

(d) A statue is known to have a mass of 88.2136… kg. When placed on a set of bathroom scales, it shows 87.8 kg. What is the error?

(e) A dog has a mass of 7.214… kg, but the weighing machine shows 7.4 kg. What is the error?

(f) The contents of a cereal packet have a mass of 456.225… g. The packet says that the contents are 450 g. What is the error?

Accuracy

If we say that a measurement is **accurate**, that means that the error is small. If we say it's rough or not very accurate, then that means the error might be quite large.

Precision

A builder used his tape measure to measure the length of a concrete house slab. He got 15.275 m. We say that measurement was precise to 3 decimal places or to 0.001 m.

But, if he did it three more times and got 15.272, 15.279 and 15.276, then it would seem that only the first two decimal places are reliable. So we might change our mind and say that the measurement was precise to only 2 decimal places.

Precision has two slightly different meanings. If we only measure once, it is the last decimal place that can be determined. But if we can measure more than once, then it is the degree to which we get the same result in successive measurements.

Now, the builder's mate with a newer tape measure measured the slab and got 15.466 m. Repeating it, he got 15.471 and 15.468.

So, while both sets of measurements were precise to the nearest hundredth of a metre or to 2 decimal places, they clearly weren't both accurate to 2 decimal places. When they compared tape measures, they saw that the first builder's tape was old, frayed and probably stretched. So they decided to use the measurement made with the newer tape measure and concluded that the length was most likely about 15.47 m.

So a measurement can be quite precise without being very accurate. If several measurements give quite a spread of values, but their average is near the true value, then we can say that the precision is low but the accuracy is good.

Precision indicates how repeatable the measurement is; accuracy indicates how close to the actual true value it is.

Practice

Q2 (a) Edgar weighs himself on the bathroom scales 5 times and gets 72.4 kg, 72.5 kg, 72.5 kg, 72.5 kg, 72.4 kg. His actual mass is 76.1 kg (rounded to one decimal place). Comment on the accuracy and the precision of the measurements.

(b) Shirley weighs herself on her scales and gets 93.1 kg, 91.6 kg, 94.4 kg, 93.9 kg and 95.6 kg. Her actual mass is 93.771… kg. Comment on the accuracy and precision of the measurements.

(c) Five people measure the distance between two trees and get 17.943 m, 17.68 m, 18.05 m, 17.88 m and 18.2 m. What is the precision of the measurements?

(d) Erin measures a stick at 84.57 cm. Jo measures it at 84.9 cm. The actual true length is 84.761… cm. Whose measurement was more accurate? which was more precise?

Random and systematic errors

The first builder's measurements were 15.272, 15.279 and 15.276 m. The mean of those was 15.2757. The deviations from the mean were –0.0037, 0.0033 and 0.0003. These deviations from the mean are called **random errors**. To find the average random error, we just average the random errors ignoring their signs. So the average of these random errors is (0.0037 + 0.0033 + 0.0003) ÷3 ≈ 0.0024.

But, if we assume that the actual length was about 15.47 m, then the mean was 0.1943 from the actual value. This deviation of the mean measurement from the actual value is called a **systematic error**.

Random errors occur because a measuring device varies a bit in operation or because it is not always possible to read a scale totally accurately. Systematic errors can be caused by a faulty measuring device or by using the device in a consistently improper way.

The total error of a measurement is the sum of the systematic and the random error.

Practice

Q3 (a) 4 people timed the period of a pendulum. They got 1.6 s, 1.9 s, 1.8 s and 1.8 s. The actual period was about 1.62 s. Find

(i) the systematic error

(ii) the average random error.

(b) The speed of light in a vacuum (to the nearest m/s) is 299 792 458 m/s. Physicists attempted to measure this speed by sending pulses of light between two hill tops. The results were 299 707 045, 299 698 448 and 299 702 189 m/s.

(i) What was the systematic error?

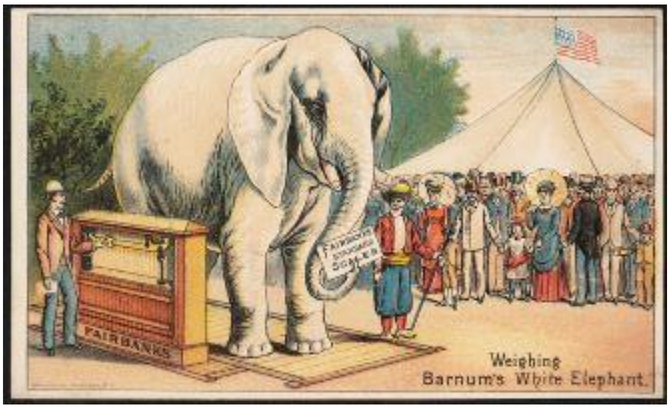
(ii) What was the average random error?

(iii) What are some possible causes of this systematic error?

(c) A sample of moon rock weighing 4.2813… kg was weighed by two different people using the same scales. If there was a systematic of –0.1223 kg and an average random error of about 0.0005 kg, what would the measurements have been?

Absolute and relative error

When weighing a 5-gram diamond, getting the error down to 0.01 g is fairly easy and fairly useful. When weighing a 4-tonne elephant, we would use a different measuring device. Getting the error down to 0.01 g would be unrealistic. Getting it down to 1 kg, would probably be possible and adequate.



But, in a sense, both measurements are equally reasonable and useful. So we often use relative error instead of absolute error. The **absolute error** is how far out the measurement is. It is expressed in terms of the units being used. The **relative error** is the absolute error divided by the actual value. It is expressed as a fraction or, more often, a percentage.

So, for the diamond,
if the absolute error is 0.01 g; then the relative error is 0.01 ÷ 5 = 0.002 or 0.2%

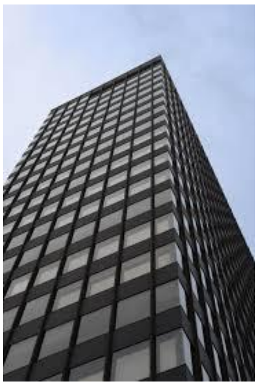
And for the elephant,
if the absolute error is 2 kg; then the relative error is 2 ÷ 4000 = 0.0005 or 0.05%

So the absolute error for the diamond is smaller, but the relative error for the elephant is smaller.

Practice

Q4 (a) A pumpkin has a true mass of about 6.285 kg. If Vi weighed it at 6.305 kg, find
(i) the absolute error
(ii) the relative error as a percentage?

(b) A song runs for 3 minutes 42 seconds. Jish timed it at 3 minutes 38 seconds. What are:
(i) the absolute error?
(ii) the relative error as a percentage?



(c) The speed of a bullet is about 345 m/s. If it is measured with an error of
–4.5%, what was the measured speed?

(d) The height of a building was measured at 92.68 m. This was an error of 0.3%. What was the actual height?

Expressing measurements with uncertainty

Suppose, when we weigh a bag of fertiliser, that it comes to 20.4 kg and we are happy that that is within 0.2 kg of the actual mass (allowing for inaccuracy of the measuring device). Then we can say that the mass is 20.4 kg ± 0.2 kg. Using the relative error rather than the absolute error, we could say that it is 20.4 kg ± 1%.

± is pronounced 'plus or minus'.

The ± 0.2 kg or the ± 1% is called the **uncertainty** associated with the measurement and indicates that we are fairly confident that the error is less than 0.2 kg (1%) and the actual mass is between 20.2 kg and 20.6 kg.

20.2 kg is called the **lower bound**, the lowest likely mass of the bag. 20.6 kg is called the **upper bound**, the highest likely mass of the bag.

Note, we don't have to be certain that it's in that range, just reasonably confident. So, to some extent, the degree of uncertainty is a matter of opinion. Someone else might have more confidence in the accuracy of the measuring device and say that the bag is 20.4 kg ± 0.1 kg i.e. 20.4 kg ± 0.5%.

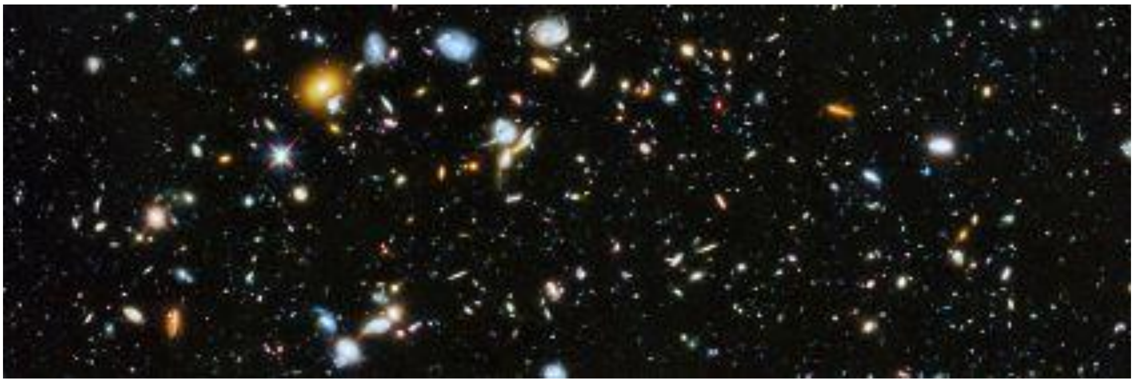
Either way, it is always possible that the true mass is less than 20 kg, though not very likely.

Practice

Q5 (a) The capacity of a container is given as 1.5 L ± 0.02 L.
What are the smallest and largest likely capacities, i.e. the lower and upper bounds of the measurement.

(b) Marjorie normally runs the 100 m in 11.3 s ± 6%. What are her slowest and fastest normal times?

(c) The age of the universe is most likely between 13.7 and 13.9 billion years. Express this as an age with an uncertainty, giving the uncertainty in billions of years and as a percentage.



(d) The width of a road is measured as 6.7 m ± 3%. Express this with the uncertainty in metres.

(e) The width of another road is measures as 8.1 m ± 0.2 m. Express this with the uncertainty as a percentage.

(f) A certain brand of light bulb is said to last for 2000 h ± 500 h. Is it possible for one to last 3000 h?

Compounding errors

Adding and subtracting measurements

Suppose we measure the four sides of a quadrilateral and find them to be 16.8 cm, 4.9 cm, 11.3 cm and 7.1 cm, each measurement with an uncertainty of ± 0.2 cm.

What would be the perimeter with its uncertainty?

Well, the most likely perimeter is the sum of the four lengths: 16.8 + 4.9 + 11.3 + 7.1 = 40.1 cm.

The uncertainty would be the sum of the four uncertainties: 0.2 × 4 = 0.8.

So the perimeter would be 40.1 cm ± 0.8 cm.

Similarly with subtraction. For instance, if a piece of string is 436 cm ± 2 cm and we cut off 100 cm ± 2 cm, the length remaining would be 336 cm ± 4 cm.

Note that we still add the uncertainties. If the string was 2 cm longer than what we measured and the bit we cut off was 2 cm shorter, then the bit left would be 4 cm shorter than our calculation.

So, when adding or subtracting measurements with uncertainties, we add or subtract the measurements, but *always add the uncertainties*.

Multiplying and dividing measurements

Suppose we measure a rectangle to be 20.3 cm ± 0.1 cm long and 13.5 cm ± 0.1 cm wide. To find its area with its uncertainty, we multiply the measurements. But now we don't multiply or add the uncertainties. In this case we have to work out the relative uncertainties and add them.

For the length, the relative uncertainty is $0.1 \div 20.3 \approx 0.005$ or 0.5%

For the width, the relative uncertainty is $0.1 \div 13.5 \approx 0.007$ or 0.7%

So the area is 274.05 cm^2 ± 1.2%. 1.2% of 274.05 ≈ 3.3 cm^2.

Uncertainties are mostly given to one significant figure, so, in this case, we would give it as 1% or 3 cm. Then, as there is 3 cm uncertainty in the measurement, the decimal places are insignificant, so we would round to the nearest square centimetre (the same place as the uncertainty). We would give the answer as 274 cm^2 ± 3 cm^2.

If we are dividing measurements, say to get the width of a rectangle from its area and length, we still add the relative uncertainties.

To find the width of a rectangle with area 10 m^2 ± 0.3 m^2 and length 4.6 m ± 0.1 m, we divide 10 by 4.6 to get 2.174. Then we work out the relative uncertainties to get 3% and 2%. Then the width is 2.174 m ± 5%. This is 2.174 m ± 0.109 m. We then adjust this to 2.2 m ± 0.1 m.

Powers

Suppose a sphere has diameter 35 cm ± 2 cm. To find the volume of the sphere, we say it is $\pi/6 \times 35^3 \approx 22\,449$ cm^3. To find the uncertainty, we find the relative uncertainty in the diameter, which is $2 \div 35 \approx 5.7\%$. Then because we are multiplying 3 lengths together, we add the uncertainties of each to get $3 \times 5.7\% = 17.1\%$.

[In other words, we multiply the percentage uncertainty by the power we are raising the measurement to.]

We get the volume is 22 449 ± 17.1%, i.e. 22 449 ± 3839.

As the uncertainty is around 4000 cm^3, we would say the volume is 22 000 cm^3 ± 4000 cm^3.

Practice

Q6 (a) The 3 sides of a triangle are measured as 32.2 cm ± 0.2 cm, 17.4 cm ± 0.2 cm, and 16.0 cm ± 0.2 cm. Give the perimeter with its uncertainty in centimetres and as a percentage.

(b) The perimeter of a block of land is known to be 110 m. Three of the four sides are measured as 37.5 m ± 1%, 12.4 m ± 2%, and 16.5 cm ± 2%. Give the length of the fourth side with its uncertainty in metres and as a percentage.

(c) A car drove at a constant speed of 80 km/h ± 3%. It drove for 4 hours ± 0.5%. Give the distance travelled with its uncertainty as a percentage and in kilometres

(d) A car drove at a constant speed of 65 km/h ± 2 km/h. It drove for 2.4 hours ± 0.2 h. Give the distance travelled with its uncertainty as a percentage and in kilometres.

(e) Cecelia ran the 1500 m in 4 min 38.2 s ± 1 s. Find her average speed in m/s and the uncertainty as a percentage and in m/s.

(f) The diameter of a sphere is measured as 81.4 cm ± 0.5 cm. Find
(i) the volume in cm^3 with its uncertainty in cm^3
(ii) the surface area in cm^2 with its uncertainty as a percentage.

(g) A cube has a volume of 534 cm^3 ± 10 cm^3. Find the edge length with its uncertainty in cm.

Rounding

On a slightly different, but related note, we should never round numbers in a calculation until the end as rounding errors can compound and cause the final answer to lie outside the acceptable value range.

Then, when you have the final answer, make sure that it is not presented with precision greater than that allowed by the input data.

For instance, if I cut a 200 cm length of material, then cut it into 7 equal lengths, it is not appropriate to say that each length will be 200 ÷ 7 = 28.5714286 cm long. Because of its stretchiness, the 200 cm will probably have an uncertainty of about ± 1 cm and the cuts might not be exactly sevenths. So the pieces will probably also have uncertainties of around 1 cm. 29 cm would be a more appropriate answer, 28.6 cm at best.

In the same way, it would be inappropriate to give the answer to Q6g as 8.11298025 ± 0.05064282 cm. 8.11 ± 0.05 cm would be better, though the many decimal places should be kept on the calculator until the final answer is reached. Of course, you don't need to show all the decimal places in your working, but keep them on the calculator. Still, even in your working, show one or two decimal places more than you will need in order to get the final answer sufficiently precisely. Working for Q6g might look like this:

$$\begin{aligned}
\textit{Volume} &= 534\text{ cm}^3 \pm 10\text{ cm}^3 \\
&= 534\text{ cm}^3 \pm 1.873\% \\
\textit{Edge length} &= \sqrt[3]{534}\text{ cm}^3 \pm 0.6242\% \\
&= 8.11298\text{ cm} \pm 0.6242\%\text{ of }8.11298 \\
&= 8.11298\text{ cm} \pm 0.05064\text{ cm} \\
&\approx 8.11\text{ cm} \pm 0.05\text{ cm}
\end{aligned}$$

Practice

Q7 A cylinder has a diameter of 15.5 cm ± 0.3 cm and a height of 21.1 cm ± 0.2 cm. Calculate its volume with the uncertainty in cm^3. Use appropriate rounding as explained above.

Applicability of these ideas

The ideas in this module are especially important in the quantitative sciences of physics and chemistry, where measurements and understanding of their uncertainty are of vital importance.

Solve

Q51 A cake weighing 725 g ± 20 g is cut into four pieces. Each piece is 0.25 ± 0.03 of the cake. Find the lowest and highest likely mass of Saharaha's piece.

Q52 At 2 pm yesterday, 10 students used flow metres to record the rate of water flow across the middle of a weir in m/s. The results were: 13.0, 1.27, 1.28, 1.33, 1.29, 1.35, 1.31, 1.25, 1.31 and 1.32. Give the likely rate of flow with the uncertainty in m/s. Explain how you came to that conclusion.

Revise

Revision Set 1

Q61 Explain why it is impossible for the period of a pendulum to be exactly 1 second.

Q62 A laboratory weight was known to be 200 g ± 0.5 g. 4 people were asked to hold it and estimate its mass. The results were 150 g, 200 g, 240 g and 210 g.
(a) Comment on the accuracy and the precision of the estimates.
(b) Comment on the random errors and the systematic error.

Q63 A block of gold with mass 200 ounces was weighed on a set of kitchen scales at 196.2 ounces. Give the absolute error and the relative error.

Q64 The length of a tape measure is given as 200.0 cm ± 0.2%. What is the range of likely actual lengths?

Q65 The three sides of a triangle are given as 34.2 cm ± 0.4 cm, 21.8 cm ± 1 cm and 18 cm ± 5%.
(i) Find the perimeter of the triangle, giving its uncertainty in centimetres and as a percentage.
(ii) Give the lower and upper bounds for the perimeter.

Q66 A rectangle has an area of 88 cm² ± 4 cm² and a width of 7.2 cm ± 0.3 cm. Find the length, giving the uncertainty in centimetres. Give the answer to appropriate precision.

Answers

Q1 (a) yes
(b) no, because the actual mass will have more non-zero decimal places than any measuring device could determine.
(c) no, because the speed will have many (perhaps an infinite number of) decimal places and it is almost infinitely unlikely that they would all be zero.
(d) –0.4136 kg
(e) 0.186 kg
(f) –6.225 g

Q2 (a) High precision but low accuracy
(b) Low precision but high accuracy
(c) Precise to the nearest metre
(d) Jo's was more accurate; Erin's was more precise.

Q3 (a) (i) 0.155 s (ii) 0.0875 s
(b) (i) –89 897 m/s (ii) 2 990 m/s
(iii) faulty instrumentation, faulty use, measuring the speed in air rather than in a vacuum
(c) 4.154 and 4.164 kg

Q4 (a) (i) 0.020 kg (ii) 0.3%
(b) (i) 4 s (ii)
(c) 329 m/s
(d) 92.40 m

Q5 (a) 1.48 L, 1.52 L
(b) 10.6 s, 12.0 s
(c) 13.8 billion years ± 0.1 billion years, 13.8 billion years ± 0.7%
(d) 6.7 m ± 0.2 m
(e) 8.1 m ± 2.5%
(f) Yes. The uncertainty shows the likely range, not the possible range.

Q6 (a) 65.6 cm ± 0.6 cm, 65.6 cm ± 1%
(b) 43.6 m ± 0.95 m, 43.6 m ± 2%
(c) 320 km ± 3.5%, 320 km ± 11 km
(d) 156 km ± 11%, 156 km ± 18 km
(e) 5.39 m/s ± 0.4%, 5.39 m/s ± 0.02 m/s
(f) 282 000 cm^3 ± 5000 cm^3, 20 816 cm^2 ± 1.2%
(g) 8.11 cm ± 0.05 cm

Q7 40000 cm^3 ± 200 cm^3

Q51 154 g, 209 g. Do the calculations using the bounds rather than the uncertainties.

Q52 1.30 m/s ± 0.05 m/s The first reading is most likely a mistake and so is ignored.

Q61 Because the period will have a very large (possibly infinite) number of decimal places and it is essentially impossible for all these to be zero.

Q62 (a) The accuracy was high, the precision low.
(b) There were quite high random errors but essentially no systematic error.

Q63 3.8 ounces, 1.9%

Q64 199.6 cm to 200.4 cm

Q65 (i) 74.0 cm ± 2.3 cm, 74.0 cm ± 3%
(ii) 71.7 cm and 76.3 cm

Q66 12.2 cm ± 1.1 cm or maybe 12 cm ± 1 cm

Image acknowledgements

Diamond: Mary Harrsch on flickr.com
Builders: Mikael Blomkvist on Pexels
Trees: Jan Tik on Flickr
Elephant: Boston Public Library on Flickr
Building: Eric Alfaro on Pexels
Fertiliser: Wikimedia Commons
Universe: NASA, ESA, H. Teplitz and M. Rafelski … on commons.wikimedia.org
Runner: Pick Pik
Material: rawpixel.com
Weir: Pacific Southwest Forest Service, USDA on Flickr