M1 Maths

Learning by Thinking

M4-1 Length, Area and Volume 4

- converting between different units for area and volume
- volumes of pyramids, cones and spheres
- surface areas of cylinders, cones and spheres
- approximating volumes of irregular 3D shapes

Learn **Answers**

This LbT (Learning by Thinking) module is an alternative to the 'Learn' section of the normal module. It is designed to lead the student to work out the maths themselves by solving problems. Thus it contains only minimal explanations. The rationale behind the approach can be read **here**.

Learn

Converting between different units for area and volume

Area

Q1 (a) How many centimetres in a metre?

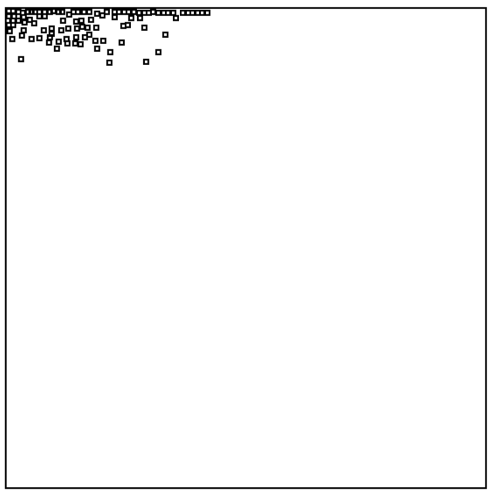
(b) Convert 5.3 m to centimetres

(c) Convert 40 cm to metres

Q2 (a) How many square centimetres (cm^2) in a square metre (m^2)?

(b) Explain why.

Hint: to the right is a diagram of a 1-m square. Inside it are drawn 100 1-cm squares. Clearly, 100 cm^2 don't make 1 m^2.



It is a very common mistake to say there are 100 cm^2 in 1 m^2. Don't make that mistake!

Q3 (a) How many square millimetres in a square centimetre?
(b) How many square millimetres in a square metre?
(c) How many square metres in a square kilometre?

The prefix kilo- means 1000. So a kilometre in 1000 m.

The prefix hecto- means 100. So a hectometre is 100 m.

A hectare is a square hectometre. The abbreviation is ha, so 15 ha means 15 hectares.

Land is often measured in hectares.

Q4 (a) How many square metres in a hectare?
(b) How many hectares in a square kilometre?

Q5 Copy and complete the following table.

cm^2	m^2	ha	km^2
	4		
		5	
80 000			
			0.002
	680		
		0.039	
500			
			3

Volume

Q6 How many 1-cm cubes would it take to fill a 1-m cube?

Hint: think about how many would be needed to make the bottom layer.

Q7 (a) How many cubic millimetres in a cubic centimetre?
(b) How many cubic metres in a cubic kilometre?

Remembering that a millilitre is the same thing as a cubic centimetre, you are now able to convert between any metric volume units – mm^3, cm^3, m^3, km^3, mL, L, kL ML etc.

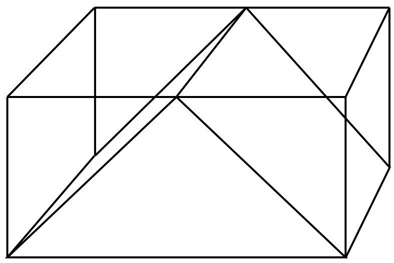
Q8 Copy and complete the following table.

L	mL	cm^3	m^3
5			
			2
		400 000	
	20		
6 000			
			0.000 004
		300	
	10 000 000 000		

Volumes of pyramids, cones and spheres

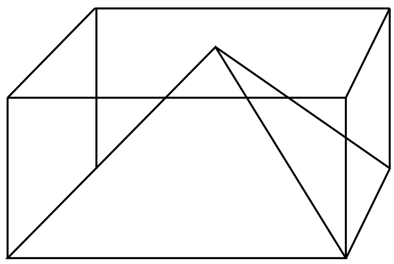
Pyramids

The diagram to the right is a triangular prism that fits snugly into a rectangular prism.



Q9 What is the fill factor, i.e. the triangular prism is what fraction of the volume of the rectangular prism?

In this next diagram the same triangular prism has had its front and back top corners shaved off to make a pyramid.

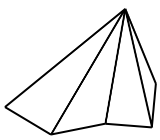


Q10 Clearly the fill factor for the pyramid is less than that for the triangular prism. Estimate it.

You should be able to tell that the fill factor for the triangular prism is ½. Clearly the fill factor for the pyramid will be less than ½. It isn't easy to tell, but the fill factor for the pyramid is exactly $^1/_3$.

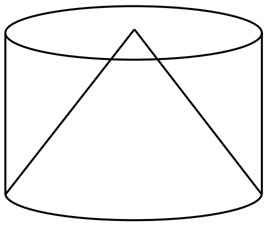
So the volume of a pyramid is *base area* × *height* × $^1/_3$.

This is the case for any pyramid, whatever the shape of the base and whether or not it is symmetrical.



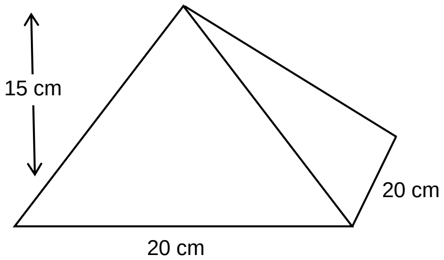
Cones

A cone is a pyramid with a circular base. So its volume is *base area* × *height* × $^1/_3$.

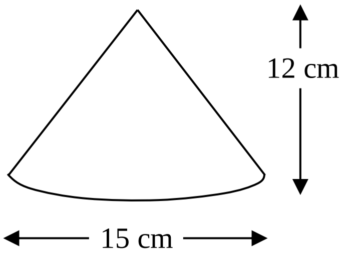


Q11 Find the volume of each of the following shapes.

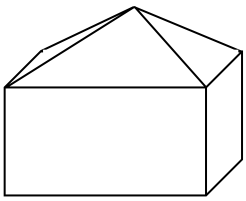
(a) square-base pyramid



(b) cone



Q12 Find the volume of this building.

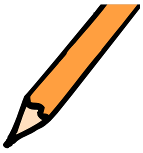


Base: 5 m by 7 m

Walls: 4 m high

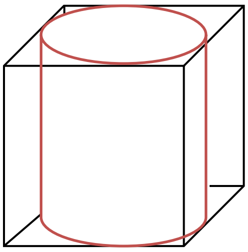
Apex 6 m above the ground

Q13 Find the volume of a round pencil 1 cm in diameter, 15 cm long, flat on one end, with the other end sharpened back 1.4 cm.



Spheres

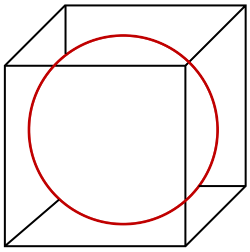
Imagine a cylinder that fits snugly inside a cube.



Q14 What is the fill factor?

Now imagine a sphere that fits snugly inside a cube.

Q15 Estimate the fill factor.



Some school maths departments have plastic cubes and spheres of the same width that can be filled with water. If you happen to have access to something like this, fill the sphere with water, then pour it into the cube to measure the fill factor.

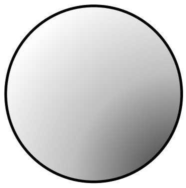
Just like the area of a circle is always about ¾ that of the enclosing square, the volume of a sphere is always about ½ that of the enclosing cube. Think of it as being $^3/_6$ of that of the cube.

And just like the ¾ was only approximate for a circle (it is actually $^{\pi}/_4$), the $^3/_6$ is only approximate for a sphere (it is actually $^{\pi}/_6$).

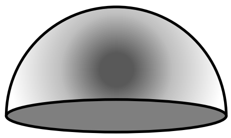
So if a sphere has a diameter of d, then the cube has a volume of d^3 and the sphere has a volume of $d^3 \times {}^{\pi}/_6$ or $^{\pi}/_6 d^3$.

Q16 Find the volume of each of the following shapes.

(a) sphere with diameter 20 cm



(b) hemisphere with radius 3 m

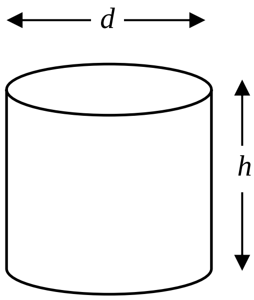


Q17 Find the volume of a cylindrical gas container with hemispherical ends, diameter 2 m, total length 10 m.

Surface areas of cylinders, cones and spheres

Cylinders

A cylinders consists of 3 faces – top, bottom and side. The top and bottom are circles.



The side is round, but imagine the cylinder is a tin of dog meat. Take off the top and bottom with a tin opener. This will leave the side, shaped as a tube. Cut the tube from top to bottom and lay it out into a rectangle.

Q18 Explain why the surface area of the cylinder is given by the formula
$A = 2 \times {}^{\pi}/_4 d^2 + \pi dh$.

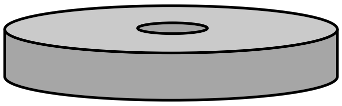
You don't really need to remember this formula and long as you can picture the cylinder being cut into three pieces and work out the area of each piece.

Q19 Find the surface area of each of the following shapes.

(a) a cylinder with diameter 10 cm and height 20 cm

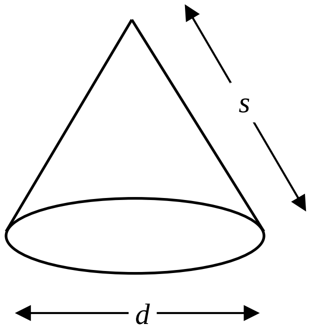
(b) a disk with diameter 12 cm and thickness 1 mm

(c) a disk with diameter 20 cm, thickness 3 cm and a 4 cm-diameter hole through the middle.

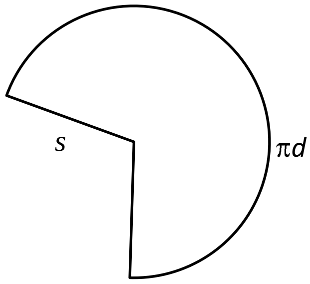


Cones

The surface of a cone is made up of two parts, the base and the side.



The side is actually a sector of a circle rolled up into a funnel shape. If you can't see this, cut a sector out of paper and roll it up to make a funnel.



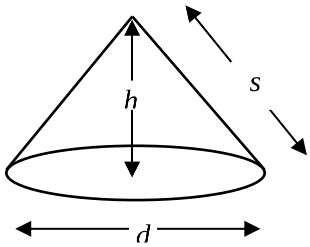
The round edge of the sector is the same length as the circumference of the base, i.e. πd. The radius of the sector is the same as the slant length of the cone, s.

Q20 Show that the total surface area is $^{\pi}/_{4}d^2 + {}^{\pi}/_{2}sd$.

This is quite an involved task. If you have trouble, this is done in the regular module M4-1 LAV 4. Look at that and then try again.

Again, you don't need to memorise this formula as long as you can work it out.

Of course, this formula assumes you know the slant height. If, instead, you know the vertical height, you can use Pythagoras to get the slant height.



Q21 Find the surface area of each of the following shapes.

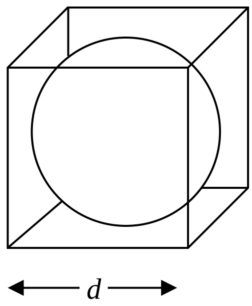
(a) a cone with base diameter 10 cm and slant height 12 cm

(b) a cone with base radius 4 cm and slant height 5 cm

(c) a cone with base diameter 10 cm and vertical height 12 cm

Spheres

Again, imagine the sphere tightly enclosed in a cube.



Just as the volume of the sphere is $\pi/6$ times that of the cube, the surface area of the sphere is $\pi/6$ times that of the cube.

So, if the sphere has a diameter of d, then the cube has a surface area of $6d^2$ and the sphere has a surface area of $6d^2 \times \pi/6$. This can be simplified to πd^2.

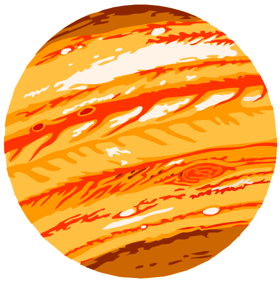
You might have noticed a pattern here: the circumference and area of a circle are both $\pi/4$ times that of the enclosing square; the surface area and volume of a sphere are both $\pi/6$ times that of the enclosing cube.

Q22 Find the surface area of each of the following shapes.

(a) a sphere of diameter 3 m

(b) a sphere with radius 20 cm

(c) a hemisphere (half a sphere) with radius 45 cm



Round things – Summary

Q23 Copy the tables below and replace the ticks with the appropriate formulae. In the first table, write the formulae in terms of diameter, d. In the second table, write them in terms of radius, r.

In terms of *d*	Circle	Cylinder	Cone	Sphere
Circumference	✓			
Area	✓			
Surface area		✓	✓	✓
Volume		✓	✓	✓

In terms of *r*	Circle	Cylinder	Cone	Sphere
Circumference	✓			
Area	✓			
Surface area		✓	✓	✓
Volume		✓	✓	✓

Approximating volumes of irregular 3D shapes

Just as we can approximate the area of a 2D shape by fitting a rectangle around it, calculating the area of the rectangle, then approximating the fraction of the rectangle taken up by the shape (the fill factor), we can approximate the volume of any 3D shape by imagining a rectangular prism around it, calculating the volume of the prism and estimating the fraction of the prism taken up by the shape (the fill factor).

Of course, you can't draw the shape properly on paper and so you can't draw the prism around it; you have to imagine it in your head, then estimate the length, width and height, then estimate the fraction taken up by the shape. Because of this, results for 3D shapes tend to be much rougher than those for 2D shapes.

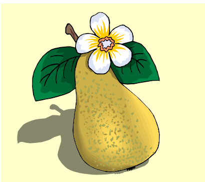
Sometimes it can be better to use a shape other than a rectangular prism. For instance, for a pile of rice, you might use a cone.

Q24 For each of these objects estimate:

(i) the dimensions of the surrounding rectangular prism (or other shape)
(ii) the fraction of the prism that would be taken up by the object
(iii) the volume of the object

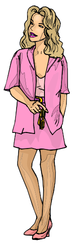
Use other things in the pictures for scale where necessary.

(a) The pear



(b) The armchair

(c) The woman



(d) The bag of fertiliser

(e) The pressed leaf

(f) The train

A variation on this method is to imagine a rectangular prism (or other shape) which is the same volume as the object you want to estimate the volume of. In places there will be space inside the rectangular prism; in other places the object will poke out from the prism.

Answers

Q1 (a) 100 (b) 530 cm (c) 0.4 m
Q2 (a) 10 000 (b) There are 100 in a row and 100 rows.
Q3 (a) 100 (b) 1 000 000 (c) 1 000 000
Q4 (a) 10 000 (b) 100
Q5

cm^2	m^2	ha	km^2
40 000	4	0.000 4	0.000 004
500 000 000	50 000	5	0.05
80 000	8	0.000 8	0.000 008
20 000 000	2 000	0.2	0.002
6 800 000	680	0.068	0.000 68
3 900 000	390	0.039	0.000 39
500	0.05	0.000 005	0.000 000 05
30 000 000 000	3 000 000	300	3

Q6 1 000 000
Q7 (a) 1 000 (b) 1 000 000 000
Q8

L	mL	cm3	m3
5	5 000	5 000	0.005
2 000	2 000 000	2 000 000	2
400	400 000	400 000	0.4
0.02	20	20	0.000 02
6 000	6 000 000	6 000 000	6
0.004	4	4	0.000 004
0.3	300	300	0.000 3
10 000 000	10 000 000 000	10 000 000 000	10 000

Q9 ½
Q11 (a) 2 000 cm^3 (b) 707 cm^3
Q12 163 m^3
Q13 11.05 cm^3
Q14 $^{\pi}/_{4}$
Q16 (a) 4189 cm^3 (b) 56.5 m^3
Q17 23.0 m^3
Q19 (a) 785 cm^2 (b) 28.3 m^2 (c) 829 cm^2
Q21 (a) 267 cm^2 (b) 113 cm^2 (c) 283 cm^2
Q22 (a) 230 cm^2 (b) 5027 cm^2 (c) 19 085 cm^2

Q23

In terms of *d*	Circle	Cylinder	Cone	Sphere
Circumference	πd			
Area	$\pi/_4 d^2$			
Surface area		$2 \times \pi/_4 d^2 + \pi dh$	$\pi/_4 d^2 + \pi/_2 sd.$	πd^2
Volume		$\pi/_4 d^2 \times h$	$1/_3 \times \pi/_4 d^2 \times h$	$\pi/_6 d^3$

In terms of *r*	Circle	Cylinder	Cone	Sphere
Circumference	$2\pi r$			
Area	πr^2			
Surface area		$2 \times \pi r^2 + 2\pi rh$	$\pi r^2 + \pi rs$	$4\pi r^2$
Volume		$\pi r^2 \times h$	$1/_3 \, \pi r^2 \times h$	$4/_3 \pi r^3$

Q24 Just check that you are in the right ballpark.

(a) 6 cm × 6 cm × 6 cm, 0.5, 100 cm^3
(b) 1 m × 1.2 m × 1.2 m, 0.4, 0.6 m^3
(c) 1.7 m × 0.4 m × 0.3 m, 0.3, 0.06 m^3
(d) 0.7 m × 0.5 m × 0.3 m, 0.8, 0.08 m^3
(e) 15 cm × 15 cm × 0.05 cm, 0.7, 8 cm^3
(f) 200 m × 3 m × 4 m, 0.8, 2000 m^3