M1 Maths

M2-3 Length, Area and Volume 2

- perimeters of circles and sectors of circles
- approximating areas of non-rectangular shapes using an enveloping rectangle
- areas of parallelograms, trapeziums, triangles, circles and sectors of circles
- surface areas (flat faces only)
- volumes of prisms and cylinders

Summary Learn Solve Revise Answers

Summary

The perimeter (circumference) of a circle is given by: *circumference* = *diameter* × π. The perimeter of a sector can be calculated by taking the appropriate fraction of the circle's circumference and adding the two radii.

To find the approximate area of a non-rectangular 2D shape, we can draw a rectangle around it, calculate the area of the rectangle, estimate the fraction of the rectangle taken up by the shape and calculate that fraction of the area of the rectangle.

Area formulae for common shapes are:

Rectangle:	*Area* = *length* × *width*
Parallelogram:	*Area* = *length* × *width*
Trapezium:	*Area* = *average length* × *width*
Triangle:	*Area* = ½ × *length* × *width* (or *Area* = ½ *base* × *height*)
Circle:	$Area = {}^{\pi}/_{4} \times diameter^2$

To find the surface area of a 3D shape, add the areas of all the faces.

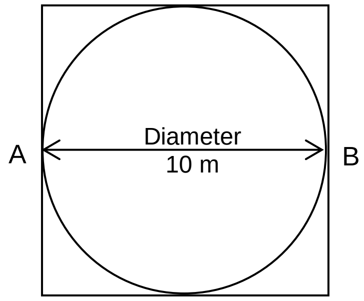
The formula for the volume of a prism or cylinder is *Volume* = *base area* × *height*.

Learn

Circumference of Circles

In Module M1-4 (Length, Area and Volume 1), you learnt how to find the perimeter of a polygon: you just add up all the side lengths. The perimeter of a circle with a known diameter can't be done this way because we don't know the side length. We use a different method.

The perimeter of a circle has a special name – **circumference**. So the circumference of a circle just means its perimeter or the distance around the outside of it. It is actually quite easy to work out the circumference of a circle if you know its diameter.



Imagine a circle with a **diameter** of 10 metres. [Diameter is the width of the circle or the distance from one side of it to the other.]

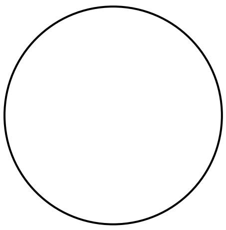
Imagine drawing a square tightly around it.

The side lengths of the square would be the same as the diameter of the circle. Can you see why? So each side of the square would be 10 m. The distance around the square would be 40 m.

So to walk from point A around the square back to point A would be a distance of 40 m. To walk from point A straight across the middle of the circle to point B and back to point A would be 20 m (twice the diameter). To walk from point A around the circle back to point A would obviously be longer than straight across the middle and back, but it would be less than going right around the square (because it would be cutting the corners).

So the distance around the circle would be more than 20 m and it would be less than 40 m. It is in fact about 30 m. In other words, the distance around the circle would be more than twice the diameter, but less than 4 times the diameter – it would be about 3 times the diameter.

This same argument would work for any circle, whatever its diameter. So the circumference of any circle is about 3 times the diameter. To calculate the circumference, just multiply the diameter by 3. A circle with a diameter of 6 cm would have a circumference of about 18 cm.



circumference ≈ diameter × 3

π

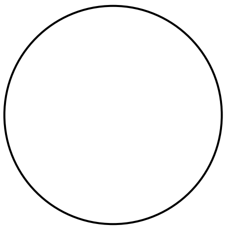
This is near enough for most practical purposes, like if you wanted to know how many bricks to buy to go around the edge of a round fish pond 2 m in diameter. But sometimes we need to be more accurate. To get a more accurate circumference, we have to multiply the diameter by 3.14. So a fish pond with a 2 m diameter actually has a circumference closer to 6.28 m than to 6 m.

3.14 is pretty close, but still isn't exactly accurate. 3.1415926 is even more accurate. Even that isn't exact though. In fact, we could keep putting more and more decimal places on the number and it would keep getting more and more accurate, but it would never be exact. The decimal places on the number go on for ever without ever stopping and without ever repeating. The first few hundred places are:

3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034
825342117067982148086513282306647093844609550582231725359408128481117450284102701938521105
559644622948954930381964428810975665933446128475648233786783165271201909145648566923460348
610454326648213393607260249141273724587006606315588174881520920962829254091715364367892590
360011330530548820466521384146951941511609433057270365759591953092186117381932611793105118
548074462379962749567351885752724891227938183011949129833673362440656643086021394946395224
737190702179860943702770539217176293176752384674818467669405132000568127145263560827785771
342757789609173637178721468440901224953430146549585371050792279689258923542019956112

Because the number cannot be written fully and exactly in this form without an infinite amount of paper, we give the number a name and just write the name. We call it π. π is a Greek p and is called pi (pronounced the same as ‘pie’ in ‘apple pie’).

So we can say that to find the circumference of a circle, we multiply the diameter by π. We often use 3.14 for the value of π, though 3 might do if we only need a rough answer.



circumference = diameter × π

Some calculators have a button for π which will give it to several decimal places. If yours doesn’t, just use 3.14.

Practice

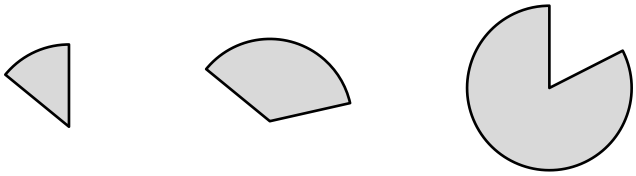
Q1 Find the approximate circumference of the following circles
- (a) one with a diameter of 5 cm
- (b) one with a diameter of 4 m
- (c) one with a diameter of 1.2 km
- (d) one with a radius of 5 cm (The radius is distance from the centre to the edge of a circle, so the diameter is twice the radius – just multiply the radius by 2.)

Q. What do you get if you divide the circumference of an apple by its diameter?

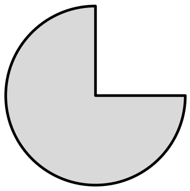
A. An apple π.

Sectors of circles

A sector of a circle is a fraction of a circle cut like a cake:

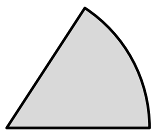


Once you know how to find perimeters of polygons and circles, finding perimeters of sectors of circles is just common sense. For instance, suppose a pizza has a diameter of 40 cm; what would be the perimeter of ¾ of the pizza?



Well, the perimeter of the whole pizza would have been $40 \times \pi$, which is 125.6 cm. The round outer edge of the ¾ pizza would be ¾ of 125.6, which is 94.2 cm. The two straight sides are each 20 cm (half the diameter). This makes a total perimeter of $94.2 + 20 + 20 = 134.2$ cm.

Suppose the angle of the sector is 50° like this:



Each degree is $^{1}/_{360}$th of the circle, so 50° is $^{50}/_{360}$ of the circle.

If the radius of the circle is 20 cm (so the diameter is 40 cm and the circumference is 125.6 cm), then the curved side of the 50° sector is $^{50}/_{360}$ of 125.6.

$^{1}/_{360}$ of 125.6 is $125.6 \div 360 = 0.3489$ cm

$^{50}/_{360}$ is $0.3489 \times 50 = 17.44$ cm.

Along with the two radii (straight sides) the perimeter is $17.44 + 20 + 20 = 57.44$ cm.

Practice

Q2 Find the perimeters of the following:

(a) ¾ of a pizza with 32 cm diameter

(b) half a pizza with a radius of 20 cm

(c) ¼ of a pizza with diameter 24 cm

(d) $^{3}/_{8}$ of a pizza with a 36 cm diameter

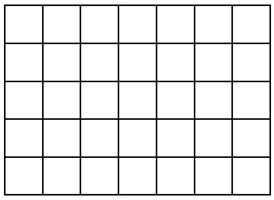
(e) a 60° sector of a circle with radius 10 cm

(f) a 145° sector with radius 6 cm

(g) a 310° sector of a circle with diameter 3 m

(h) a 4° sector with diameter 1.16 m

Finding approximate areas of non-rectangular shapes



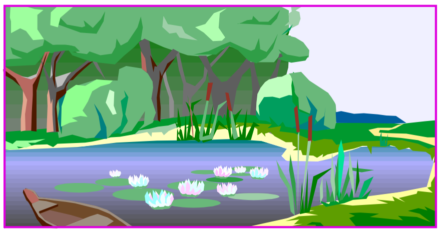
You should remember from Module M1-4 that the area of a shape is the number of unit squares it takes to cover it. For instance, the area in cm^2 is the number of 1 cm by 1 cm squares it takes to cover it. You should also remember that the area of a rectangle is found by multiplying the length by the width:

Area = *length* × *width*.

For instance, to work out the area of this rectangle we do:

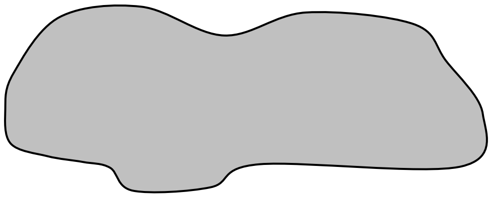
Area = *length* × *width* = 10 × 4 = 40 m^2.

But you do not know how to calculate the area of any other shape.



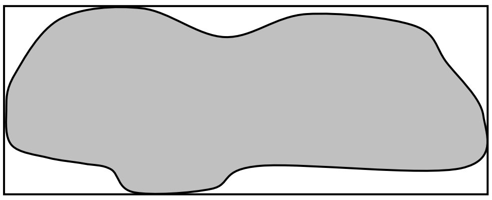
Suppose that you needed to find the area of an irregular shape, like a lake for instance, or a coloured patch on a tile you were designing.

Let's say you need to find the area of this shape.



There is no way of calculating the area of this shape exactly. But you can approximate it using the following technique.

1. Draw a rectangle which the shape just fits into.



2. Measure the length and width of the rectangle and calculate its area. This one is 7.2 cm long and 2.8 cm wide. So its area is 7.2 × 2.8 = 20.16 cm^2. As the length and width are only accurate to about 0.1 cm, the area is probably only accurate to the nearest cm^2. So let's say the area is 20 cm^2.

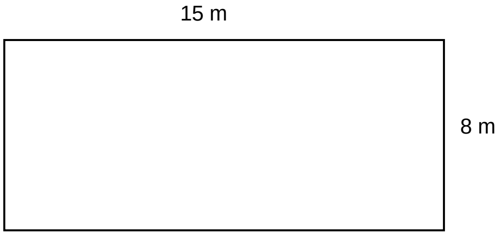
3. The original shape is smaller than the rectangle. By looking at it, we can estimate that it takes up about 80% of the rectangle. So its area is about 80% of the area of the rectangle. 80% of 20 cm^2 is 16 cm^2. So the area of the shape is about 16 cm^2.

The same technique can be used for any shape, including triangles, parallelograms and circles.

Practice

Q3 Calculate the areas of these rectangles.

(a)



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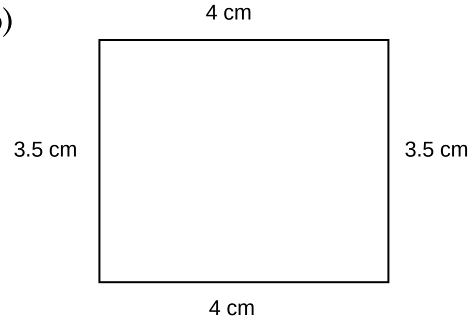
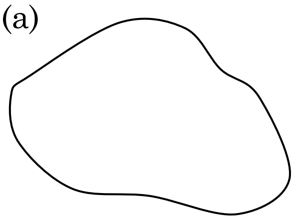


Q4 Draw shapes on paper something like these. Then draw a rectangle around each to find its approximate area.

(a)



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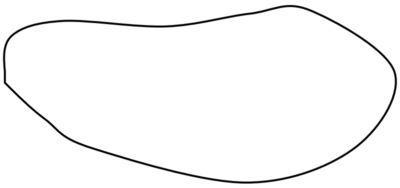

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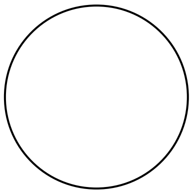
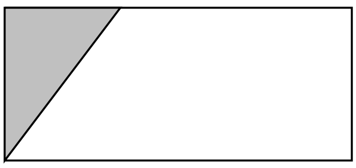


Areas of Parallelograms

Areas of parallelograms can be worked out the same way, but there is an easier way that will give an exact answer. Take a parallelogram.

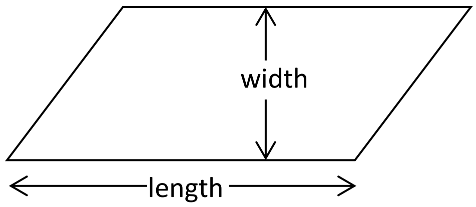
We can divide it into two pieces like this:

Then we can cut the triangle off the right end and stick it on the left end like this to make a rectangle.



The rectangle has the same area as the original paralleogram because we haven't added or removed anything. The area of the rectangle is the length of the rectangle multiplied by the width of the rectangle. This is the length of one side of the parallelogram multiplied by the width of the parallelogram.

In other words we multiply the length by the width, but remembering that the length and width are measured as in the diagram below, perpendicular to (at 90° to) each other.



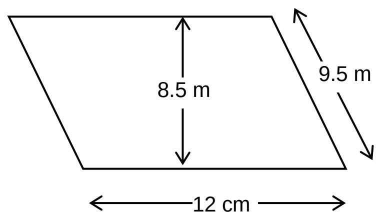
In this example, the length is 4.7 cm and the width is 2.0 cm.
So the area is $4.7 \times 2.0 = 9.4$ cm^2.

So, for a parallelogram, $Area = length \times width$.

Practice

Q5 Find the areas of the following parallelograms.

(a)



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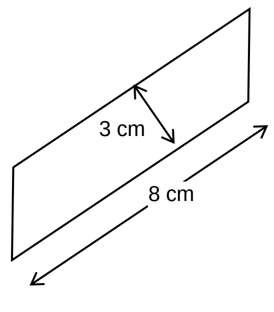

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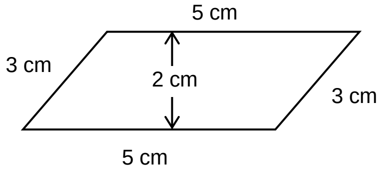

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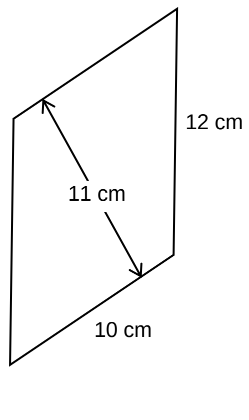
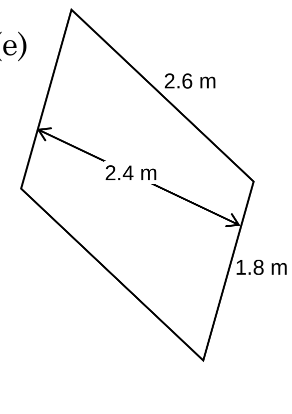


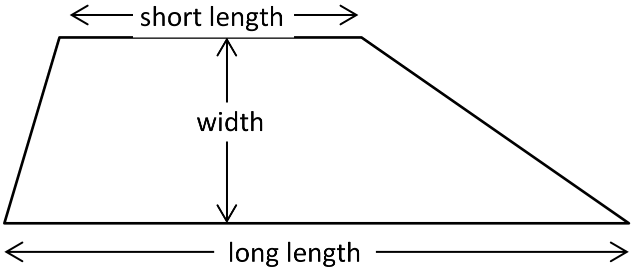
(e)



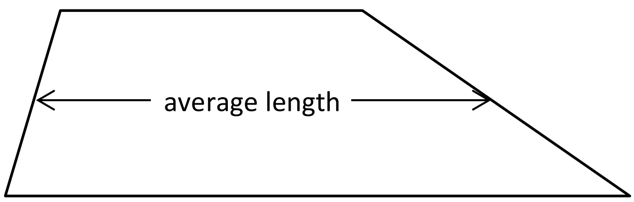
Areas of Trapeziums

The area of a trapezium can be worked out exactly too.

Consider the trapezium below. If we count the top and bottom sides as the length, then it has a short length (top) and a long length (bottom). As always, the width is perpendicular to the length.

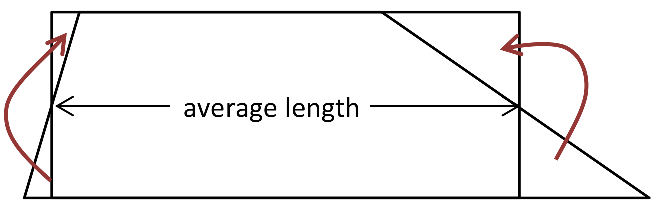


Half way up the trapezium, the length will be the average of the long length and the short length. We call this the average length. To find the average length, we add the short length and the long length, then divide the result by 2 (as you learnt in S1-2 – Date Summary).



The area of the trapezium is then the average length multiplied by the width.

If this isn't obvious, then consider cutting off the ends of the bottom half of the trapezium and adding them to the top half like this:



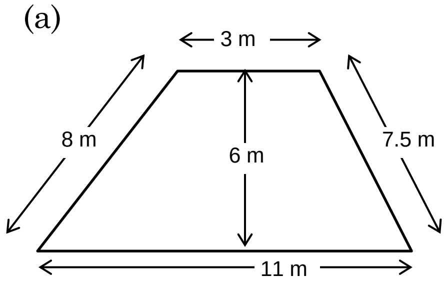
The shape has been rearranged to a rectangle with area $average\ length \times width$.

So, for a trapezium, $Area = average\ length \times width$.

Practice

Q6 Find the areas of the following trapeziums using the measurements given.

(a)



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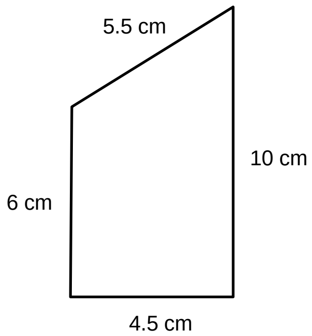

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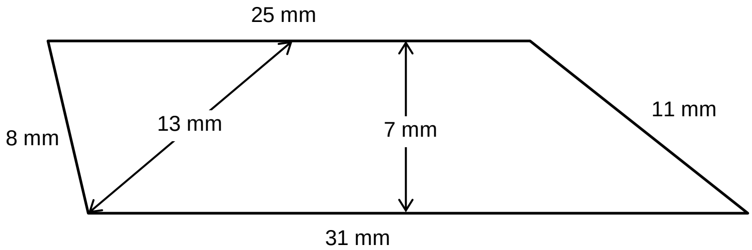
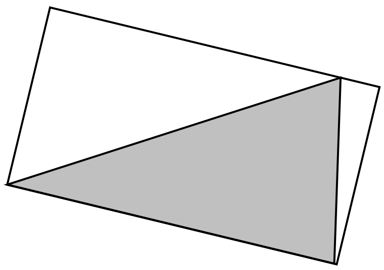
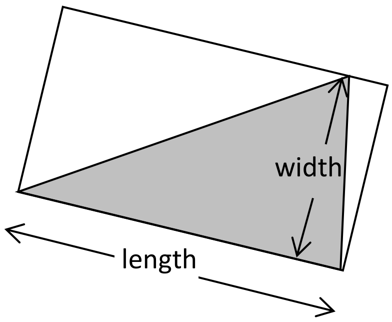


Areas of Triangles

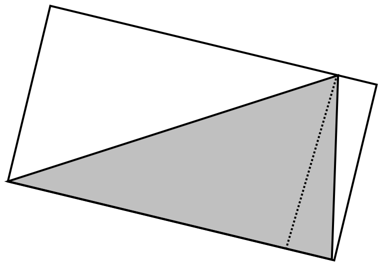
A rectangle can be drawn around any triangle such that one side of the triangle coincides with one side of the rectangle and the opposite vertex of the triangle is on the opposite side of the rectangle, like this:



This way, the length of the triangle is equal to the length of the rectangle and the width of the triangle is equal to the width of the rectangle.



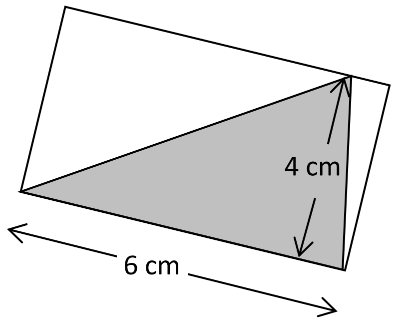
Then, the area of the triangle will be half the area of the rectangle. We can see this by adding a dotted line like this:



The dotted line divides the rectangle into two smaller rectangles. Each of these is half shaded, half unshaded. So there are equal amounts of shaded and unshaded area in the big rectangle. In other words, half the rectangle is shaded. This, of course, means that the triangle is half the area of the rectangle.

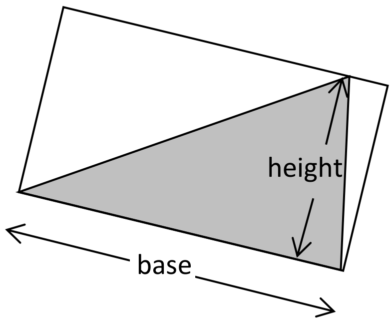
The area of any triangle is half the area of the rectangle that fits around it and so is always half of the length × width for the triangle.

The triangle to the right is 6 cm long by 4 cm wide. The rectangle is 6 cm by 4 cm, so its area is 24 cm². The area of the triangle is therefore ½ of 24 cm² which is 12 cm².



For a triangle, *Area* = ½ *length* × *width*.

It is common to call the length of a triangle *the base* and to call the width *the height*.

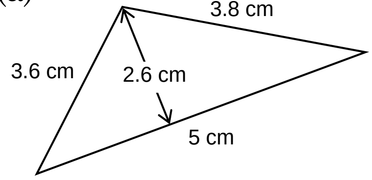


So we can say, for a triangle, *Area* = ½ *base* × *height*. Both are equally correct. Don't forget, though, that the height must always be measured perpendicular to the base.

Practice

Q7 Find the areas of the following triangles using the measurements given.

(a)



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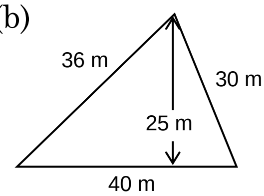

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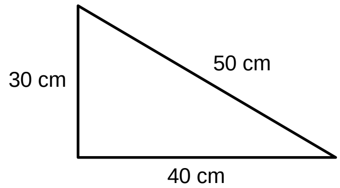

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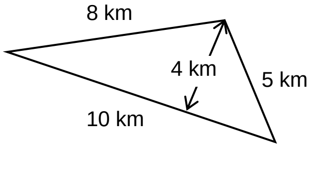

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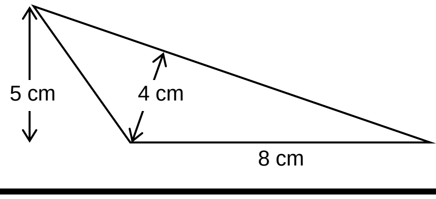
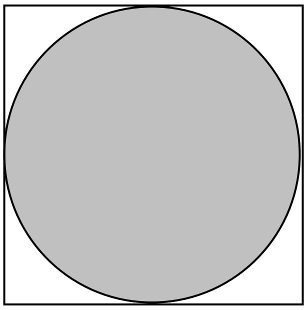


Areas of Circles

A rectangle can be drawn snugly around a circle – like this:



The length and width of the rectangle will be the same as the diameter of the circle. The rectangle will be a square because the length and width are the same. [Remember that a square is a type of rectangle.]

If the diameter of this circle is 2.4 cm, then the area of the square will be $2.4 \times 2.4 = 5.8$ cm^2.

We can estimate the fraction of the square taken up by the circle. It is about ¾. In fact it doesn't matter how big the circle is, if we draw a square snugly around it, the circle will take up about ¾ of the square.

So all we need do is find ¾ of 5.8. ¼ of 5.8 is 1.45, so to get ¾, we multiply 1.45 by 3 and get 4.35. So the area of the circle is about 4.4 cm^2.

Remember how when calculating circumferences of circles, we multiply the diameter by 3 to get a rough answer and π (3.14) to get an exact answer? Well the same happens with areas. 3 quarters of the area of the square is a rough answer, but the exact answer is π quarters of the area of the square. So, to do the calculation more accurately, we would divide 5.8 by 4 like we did before to get 1.45. But then we would multiply that by 3.14 to get 4.6 cm^2.

So for a circle, $Area = {}^{\pi}/_{4} \times diameter \times diameter$, or $Area = {}^{\pi}/_{4} \times diameter^2$.

You may see another formula for the area of a circle: $Area = \pi \times radius^2$. The radius is half the diameter, so $radius^2$ is ¼ of $diameter^2$. So there is no need to divide by 4. Both formulae will always give the same result.

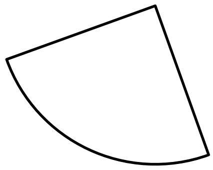
Sectors of Circles

The area of a part of a circle is easy to find. If a whole circle has an area of 60 cm^2, then

½ of it will have an area of ½ of 60, i.e. 30 cm^2;

$^3/_5$ of it will have an area of $^3/_5$ of 60, i.e. 36 cm^2;

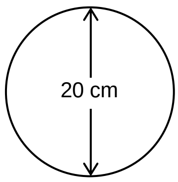
a 65°sector will have an area of $^{65}/_{360}$ of 60 and so on.



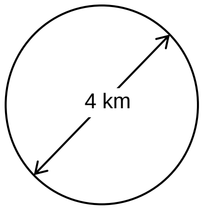
Practice

Q8 Find the areas of the following shapes using the measurements given.

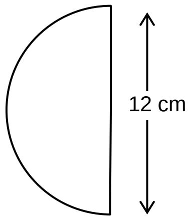
(a)



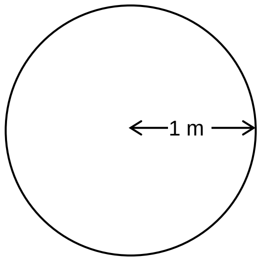
(b)



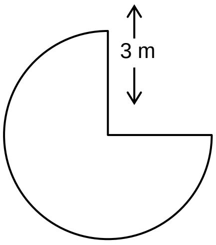
(c)



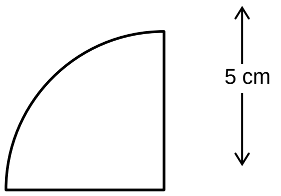
(d)



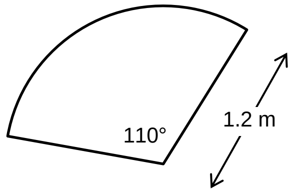
(e)



(f)



(g)



To summarise, the area formulae for the shapes we have looked at are:

Rectangle: $Area = length \times width$

Parallelogram: $Area = length \times width$

Trapezium: $Area = average\ length \times width$

Triangle: $Area = \frac{1}{2} \times length \times width$ (or $Area = \frac{1}{2}\ base \times height$)

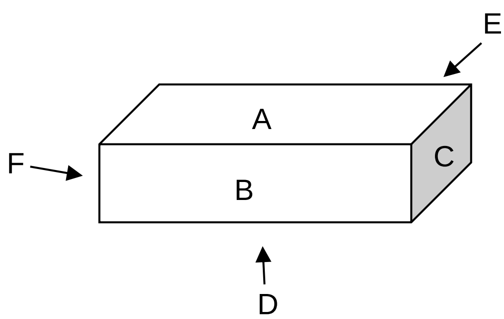
Circle: $Area = {}^{\pi}/_{4} \times diameter^2$

Surface Areas

The area of a 2D shape is the number of unit squares needed to cover it. The surface area of a 3D shape is the number of unit squares needed to cover its whole surface, i.e. all its faces.

Calculating surface areas is easy. You just work out the area of each face and then add them all up. Of course, to be able to calculate the areas of the faces, they must be rectangles, parallelograms, trapeziums, triangles, or circles.

It can help to draw a diagram and label the faces to keep track of which ones you have done. For example, suppose we wanted to know the surface area of a 10 cm by 5 cm by 2 cm rectangular prism. We can draw it like this and label the faces A to F.



D is the bottom (the face opposite A), E is the back (the face opposite B), and F is the left side (the face opposite C). We list the areas of the faces as follows:

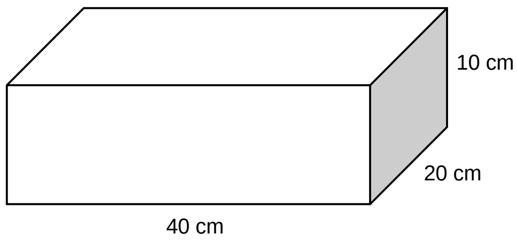
A: $10 \times 5 = 50$
B: $10 \times 2 = 20$
C: $5 \times 2 = 10$
D: $10 \times 5 = 50$
E: $10 \times 2 = 20$
F: $5 \times 2 = 10$
Total $= 160$

Adding these up we get 160 cm^2. So the surface area of the prism is 160 cm^2.

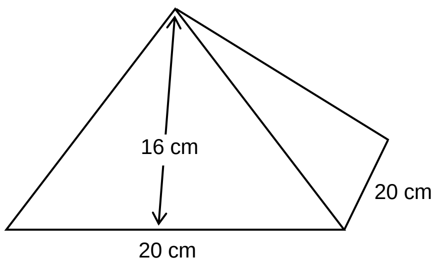
Practice

Q9 Calculate the surface areas of the following shapes.

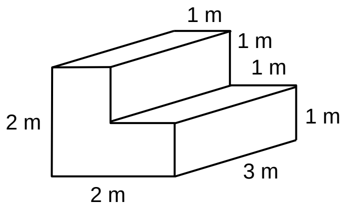
(a)



(b) (square-based pyramid)

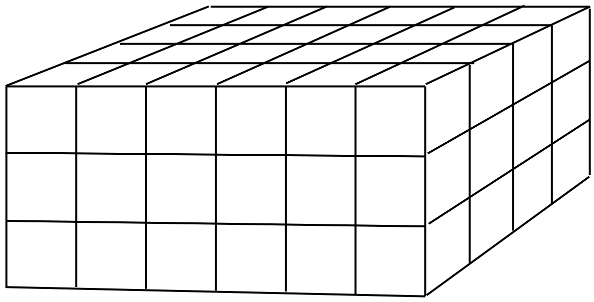


(c)



Volumes of Prisms and Cylinders

In Module M1-4 you learnt how to find the volume of a rectangular prism. Volume is the number of 1 cm cubes it takes to make the prism. The prism to the right is 6 cm by 4 cm by 3 cm. The individual cubes that it would take to make it are shown.



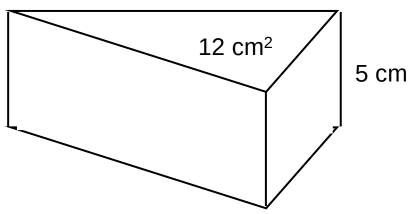
The prism has 24 cubes in each layer (you can count on the top – 4 rows of 6) and it has 3 layers. So the total number of cubes is 72 and the volume is 72 cm^3.

The number of cubes in a layer is the same as the area of the layer. This is because the top of each cube is a 1 cm square. Also, the number of layers is the same as the height of the prism. This is because each layer is 1 cm high.

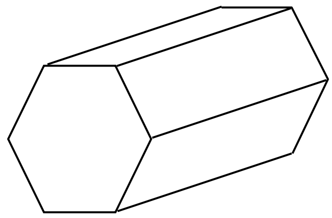
So another way of looking at the volume is that it is the area of the base or top of the prism multiplied by its height.

This way of looking at the volume of a prism has the advantage that it can be used for prisms of any shape. The volume of any prism is equal to the area of the base multiplied by the height.

So, if we had a triangular prism like this where the area of the top or base is 12 cm^2 and the height is 5 cm, then the volume is 60 cm^3.

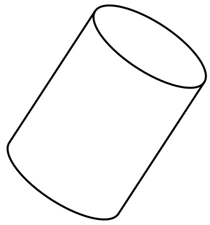


Of course a prism is a 3D shape that has the same size and shape right along its length. In other words, if we sliced it like a loaf of bread, all the slices would be the same. In the example above, we would slice the prism into horizontal slices, but prisms can be standing other ways. In the hexagonal prism below, the slices would have to be vertical for them all to be the same.



In this case, rather than talk about the base of the prism and its height, we might talk about its end and its length. In this case the volume would be the area of the hexagonal end multiplied by the length.

A cylinder is a circular prism. The volume is the area of the circular end multiplied by the length. So, for the cylinder to the right with diameter 6 cm and height 10 cm, we work out the volume like this:



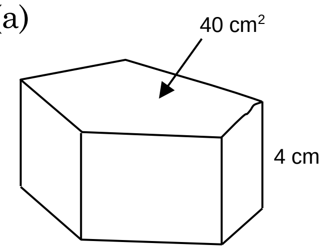
The diameter is 6 cm, so the area of the end is $3.14 \div 4 \times 6 \times 6$, which is 28.26.

The volume is 28.26×10, which is 282.6 cm^3.

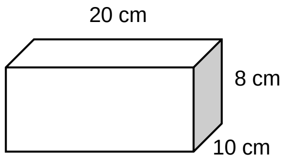
Practice

Q10 Calculate the volumes of the following shapes.

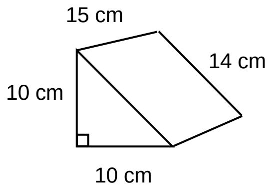
(a)



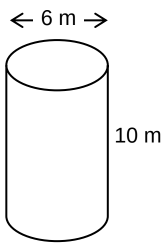
(b)



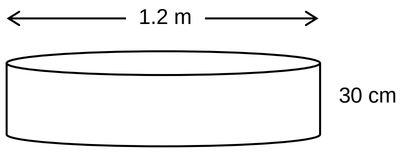
(c)



(d)

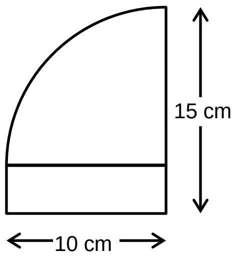


(e)

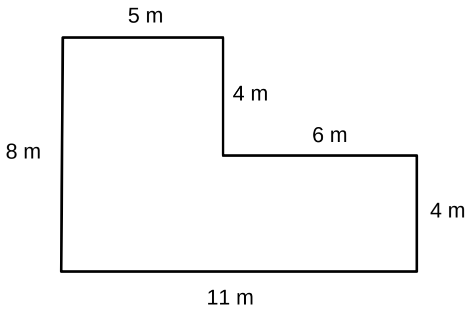


Solve

Q51 Find the perimeter of this shape made from a quarter circle and a rectangle:

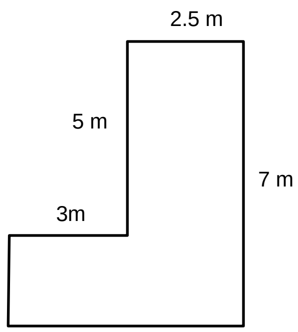


Q52

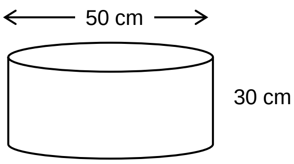


Find the floor area of this L-shape room.

Q53 Find the floor area of this L-shape room:



Q54



Find the surface area of this cylinder:

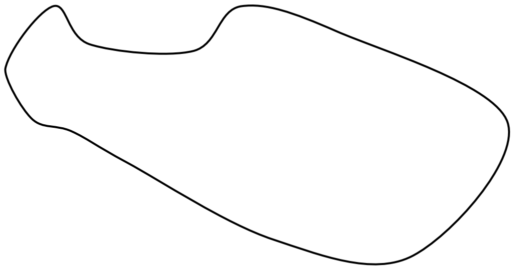
Revise

Revision Set 1

Q61 Find the circumference of a circle with diameter 12 cm.

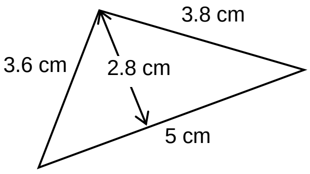
Q62 Find the perimeter of a 127° sector of a circle with radius 22 cm.

Q63 Draw an approximation to this shape, then draw a rectangle around it to find its approximate area.

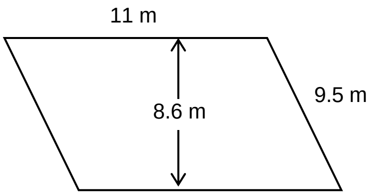


Q64 Find the areas of the following shapes using the measurements given.

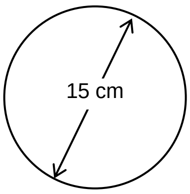
(a)



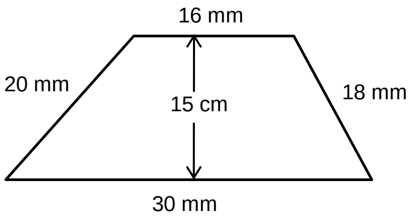
(b)



(c)

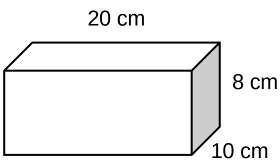


(d)

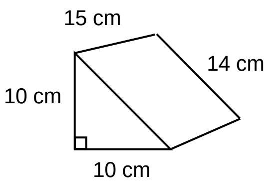


Q65 Calculate the surface areas of the following shapes.

(a)

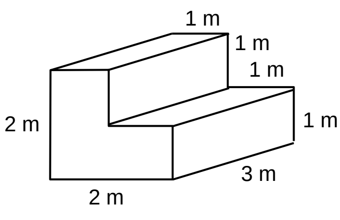


(b)

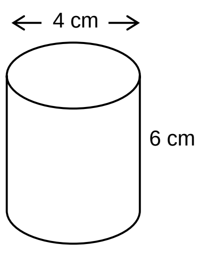


Q66 Calculate the volumes of the following shapes.

(a)



(b)



Answers

Q1	(a) 15.7 cm	(b) 12.56 m	(c) 3.768 km	(d) 31.4 cm
Q2	(a) 107 cm	(b) 103 cm	(c) 42.8 cm	(d) 78.4 cm
	(e) 30.5 cm	(f) 27.2 cm	(g) 11.1 m	(h) 1.20 m
Q3	(a) 120 m^2	(b) 14 cm^2		
Q5	(a) 102 cm^2	(b) 24 cm^2		
	(c) 10 cm^2	(d) 110 cm^2	(e) 4.32 m^2	
Q6	(a) 42 m^2	(b) 36 cm^2	(c) 196 mm^2	
Q7	(a) 6.5 cm^2	(b) 500 m^2	(c) 600 cm^2	
	(d) 20 km^2	(e) 20 cm^2		
Q8	(a) 314 cm^2	(b) 12.56 km^2	(c) 56.52 cm^2	(d) 3.14 m^2
	(e) 21.2 m^2	(f) 19.63 cm^2	(g) 1.38 m^2	
Q9	(a) 2 800 cm^2	(b) 1 040 cm^2	(c) 30 m^2	
Q10	(a) 160 cm^3	(b) 1 600 cm^3	(c) 750 cm^3	
	(d) 283 m^3	(e) 0.339 m^3 or 339 000 cm^3		

Q51	92.8 cm	Q52 64 m^2	Q53 23.5 m^2	Q54 4112 cm^2

Q71	37.68 cm			
Q72	92.8 cm			
Q74	(a) 7.0 cm^2	(b) 94.6 cm^2	(c) 176.6 cm^2	(d) 345 cm^2
Q75	(a) 880 cm^2	(b) 610 cm^2		
Q76	(a) 9 m^3	(b) 75.4 cm^3		