M1 Maths

Learning by Thinking

M2-3 Length, Area and Volume 2

- perimeters of circles and sectors of circles
- approximating areas of non-rectangular shapes using an enveloping rectangle
- areas of triangles, circles, parallelograms, trapeziums and sectors of circles
- surface areas (flat faces only)
- volumes of prisms and cylinders

Learn Answers

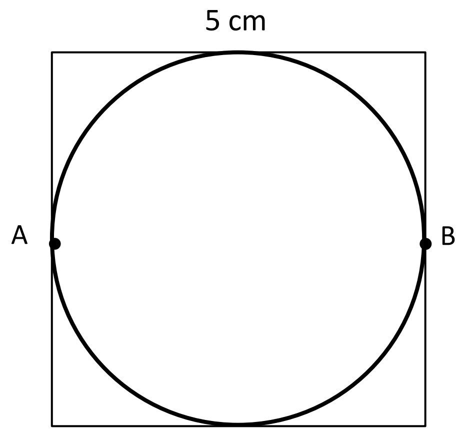
This LbT (Learning by Thinking) module is an alternative to the ‘Learn’ section of the normal module. It is designed to lead the student to work out the maths themselves by solving problems. Thus it contains only minimal explanations. The rationale behind the approach can be read **here**.

Learn

Perimeters of circles

As with polygons, the perimeter of a circle is the distance around it. The perimeter of a circle can also called the ***circumference***.

The circle below has a diameter (distance from one side to the other) of 5 cm. It is drawn inside a 5 cm square.



Q1 Imagine an ant standing at A. If it walked straight across the middle of the circle to B and then straight back to A, how far would it walk?

Q2 The same ant walks from A to B and then back to A again along the edges of the square. In other words, it walks the perimeter of the square. How far would it walk?

Q3 The same ant then walks from A to B and back to A around the perimeter of the circle. Roughly how far would it walk?

Obviously when walking around the circle, the ant walked further than going straight across the middle and back, but, because it cut the corners, not as far as walking around the square.

You should have noticed that the distance across the middle and back is twice the diameter, i.e. 10 cm; the distance around the square is 4 times the diameter, i.e. 20 cm; the distance around the circle would be somewhere between these. 3 times the diameter would be a good guess, i.e. 15 cm.

This is a good approximate value for the perimeter (circumference) of the circle. We can say that the circumference of a circle is about 3 times the diameter. The same would be true for circles of any size: the circumference is always about three times the diameter.

Q4 Find something round, e.g. a food tin, measure its diameter and measure its circumference. Is the circumference exactly 3 times the diameter?

If you measured accurately, you should have found that the circumference is a little more than 3 times the diameter. In fact, if we could measure accurately enough, the circumference would be about 3.14 times the diameter, or 3.1415926 times to be more precise. In fact, if we were totally accurate (which is of course impossible), we would find that the circumference is 3.141592653589793238462643383279… times the diameter. The … means that the decimals keep going. In fact they go on for ever. We call this number π (the Greek letter *p*, spelt ‘pi’ and pronounced the same as ‘pie’). Your calculator probably has a π button to save you punching in the number with its decimal places.

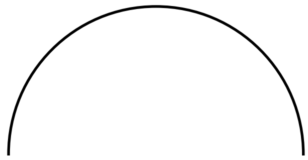
So, to find the circumference of a circle, we multiply the diameter by π.

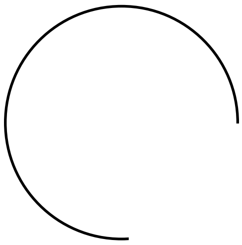
Q5 Find the circumferences of circles with diameters
(a) 10 cm (b) 3 m.

It is worth remembering for future that all measurements of perimeters, areas, surface areas and volumes of circles and spheres (anything round) will involve multiplying by π. 3 for an approximate answer will always be replaced by π for an accurate one.

Perimeters of arcs and sectors of circles

Q6 This is a semicircle (half a circle). Its diameter is 4 cm. Find the length of the curved line (arc).

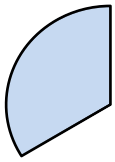


Q7 

This is ¾ of a circle with diameter 3 cm. Find the length of the curve.

Q8 Fractions of circles like this are called ***arcs***. Find the arc length of a quarter of a circle with a diameter of 40 m.

Q9 This shape is called a sector of a circle. It is $^1/_3$ of a circle. It's like a slice of pizza. Its perimeter consists of the arc plus the two straight lines (radii). The radius is half the diameter. If the radius is 10 cm, find the perimeter.



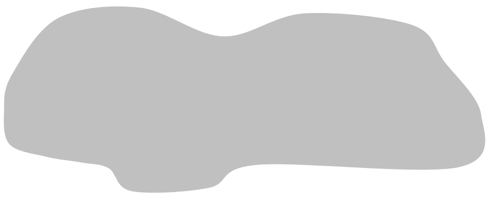
Areas of non-rectangular shapes

You should remember from Module M1-4 that the area of a rectangle can be calculated by multiplying the length by the width. We can write this as the formula

$$A = l \times w$$

where A is the area of the rectangle, l is its length and w is its width.

Q10 The picture below is of an ink stain on someone's dress. Copy the stain onto paper. Then draw a rectangle around it such that the rectangle is as small as possible, but also such that all of the stain is inside the rectangle.



Q11 Measure the length and width of the rectangle and calculate its area. [Note: your answer will depend on how big you draw the stain.]

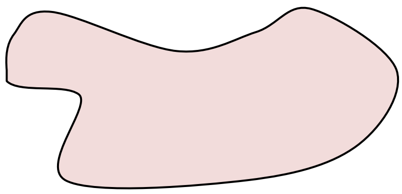
Q12 Estimate the fraction of the rectangle taken up by the stain. This is called the ***fill factor***.

Q13 Use your area of the rectangle and the fill factor to estimate the area of the stain.

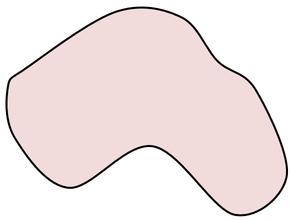
This method is called the ***envelope method*** because you are drawing a rectangular envelope around the shape.

Q14 For each of these shapes, copy the shape onto a piece of paper, then use the envelope method to estimate its area.

(a)



(b)



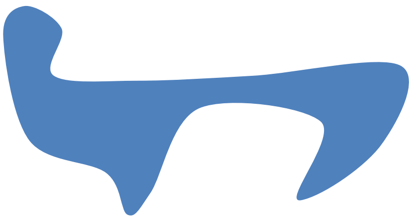
In using the envelope method, you are multiplying the length of the rectangle by its width, then multiplying by the fill factor. We can write this method as a formula:

$$A = l \times w \times f$$

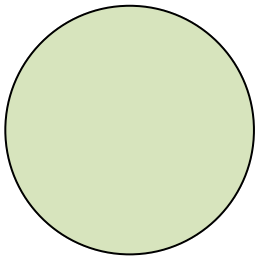
where A is the area of the shape, l is the length of the rectangle, w is the width of the rectangle and f is the fill factor.

Q15 Copy these shapes, then use the envelope method and the formula to estimate their area.

(a)

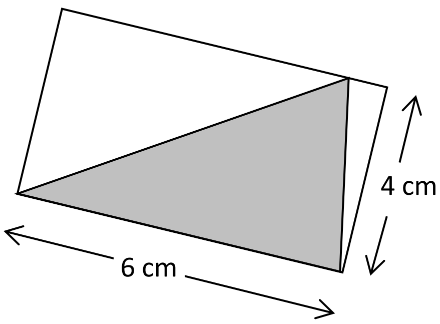


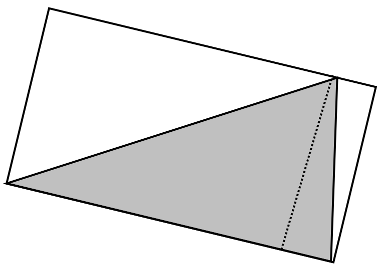
(b)



Areas of Triangles

Use the envelope method to estimate the area of this grey triangle.

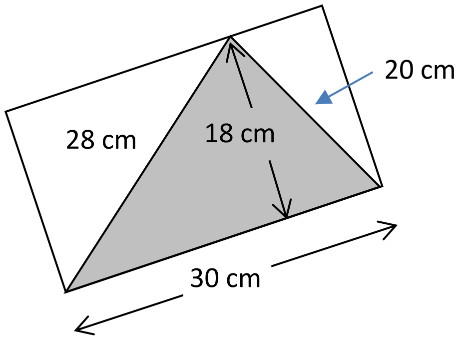




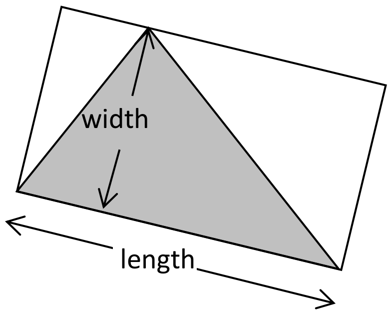
Hopefully, you estimated the fill factor to be about ½. In fact, it would be exactly ½.

If we divide the triangle into two like this it is easier to see why it would be exactly ½.

Q16 Explain why the fill factor will be exactly ½.



Q17 Use the envelope method to find the area of this triangle

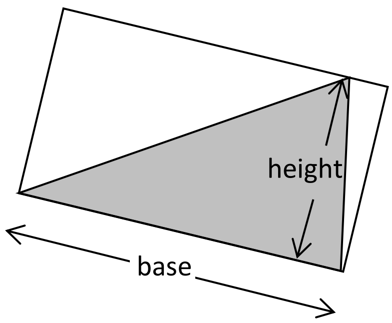


You should see that the width must be measured perpendicular to (at right angles to) the length. So, in the last question, you should have used 18 cm for the width, not 20 cm or 28 cm.

When using the formula $A = l \times w \times f$, l and w must always be measured perpendicular to each other. This is obvious for rectangles, but not so obvious for other shapes, like triangles.

For a triangle, A = *length* × *width* × ½ or *A* = *l* × *w* × ½

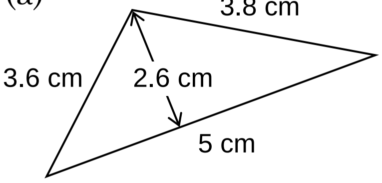
The length of a triangle is sometimes called *the base* and the width is sometimes called *the height*.



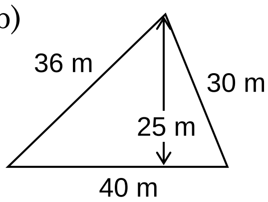
So we could say, for a triangle, *Area* = *base* × *height* × ½ .

Q18 Find the areas of the following triangles using the measurements given.

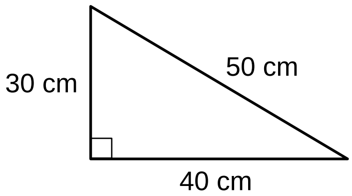
(a)



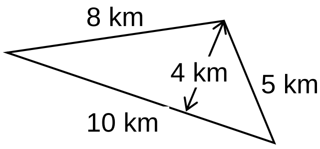
(b)



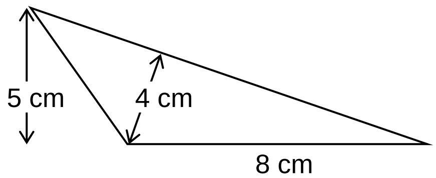
(c)



(d)



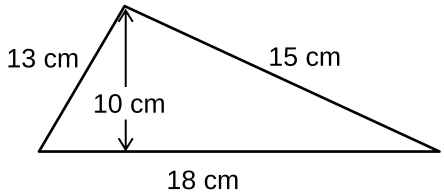
(e)



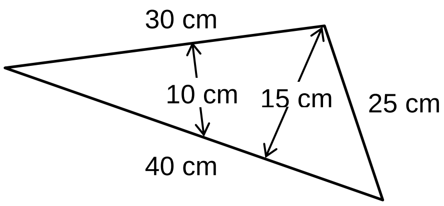
Note that, in the last triangle in the question above, the length of the top side is not given, but we can use the length of the bottom side (8 cm) for the length and the 5 cm as the width. You will see this more clearly once we have looked at parallelograms.

Q19 Find the areas of theses triangles.

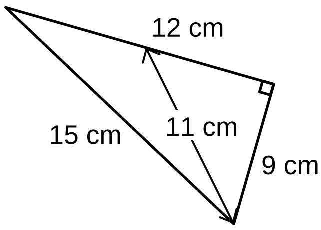
(a)



(b)

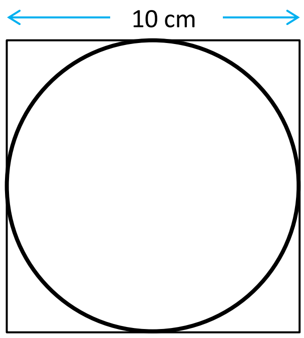


(c)



Areas of Circles

Q20 Use the envelope method to estimate the area of this circle.



You probably estimated the fill factor to be about ¾.

Remember that we found that the perimeter of the circle is about ¾ of the perimeter of the square, but to be precise, it is $^{\pi}/_{4}$. We said that the perimeters, areas and volumes of anything round all have a π in, replacing a 3 in the approximate value.

Well, this is true of the area of the circle, so, instead of ¾ the area of the square, it is $^{\pi}/_{4}$ the area of the square.

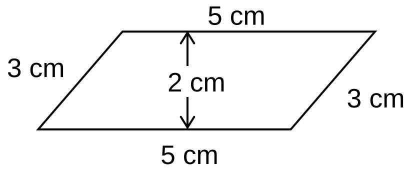
Q21 Find the more exact area of the circle above.

Q22 Find the areas of circles with
(a) diameter 4 m (b) radius 6 cm (c) perimeter 1 m

Areas of Parallelograms

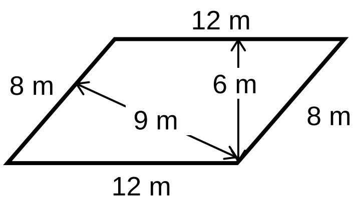
A parallelogram is a quadrilateral (4-sided shape) with opposite sides parallel (and therefore also equal).

Q23 Find the area of this parallelogram. Hint: you can cut a triangle off the right side and stick it on the left side to change it into a rectangle with the same area.



Q24 What two measurements do you need in order to calculate the area of a parallelogram?

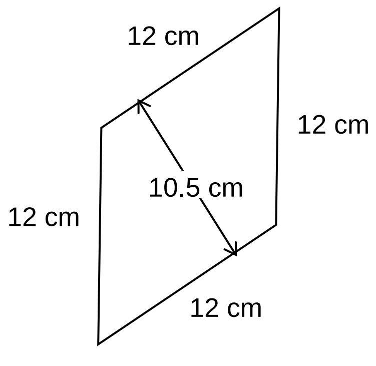
You should have found that the measurements needed are the length parallel to one of the sides and the width perpendicular (at 90°) to that side.

Q25 

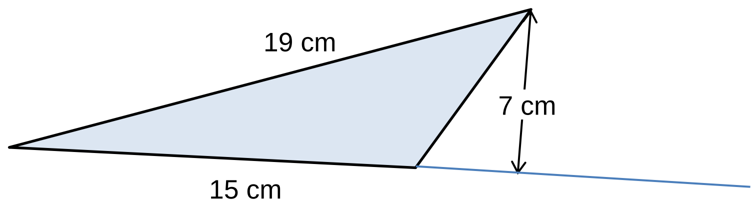
Find the area of the parallelogram to the left.

Q26 What are the two possible multiplications that could have been used to answer the last question?

Q27 Find the area of this rhombus. (A rhombus is a parallelogram with *all* sides equal.)

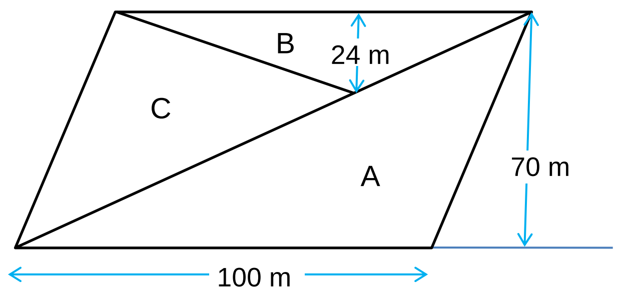


Q28 Draw another triangle the same size and shape as the blue one here so that the two triangles together form a parallelogram. Then use the diagram to explain why the area of the blue triangle is given by 15 × 7 × ½.



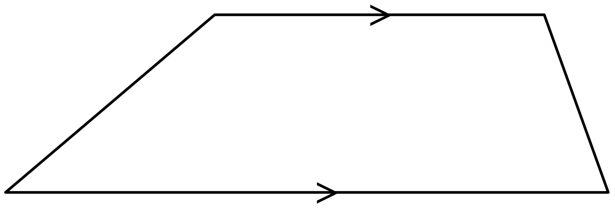
This should make the method used for Q18e a bit clearer.

Q29 The block of land below is in the shape of a parallelogram. It is to be cut into three triangular blocks, A, B and C. Find the area of each.

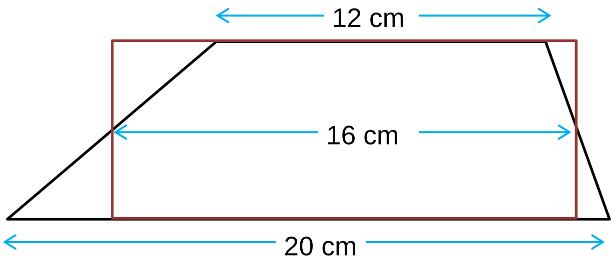


Areas of Trapeziums

A trapezium is a quadrilateral with just two sides parallel, like this:



Suppose the top of the trapezium is 12 cm long and the bottom is 20 cm long. Imagine superimposing a rectangle on it like this red one, with the length of the rectangle half-way between the length of the top of the trapezium and the length of the bottom of the trapezium, i.e. 16 cm.



Q30 Is the area of the rectangle greater than, less than or the same as the area of the trapezium?

You should have realised that it is the same because the bits of the trapezium sticking out of the rectangle would exactly fit in the gaps in the top corners of the rectangle.

So the area of the trapezium is the same as that of the rectangle with length half way between top and bottom lengths of the trapezium and width the same as the trapezium.

Q31 (a) What number is half-way between 12 and 20?

(b) What is half the sum of 12 and 20?

(c) What is $\frac{12+20}{2}$?

(d) What is the mean (average) of 12 and 20?

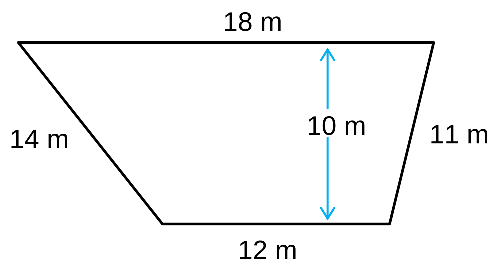
You should realise that the last 4 questions have the same answer because they are really four different ways of asking the same question.

So the area of a trapezium is the average length (measured parallel to the parallel sides) multiplied by the width measure perpendicular to the length.

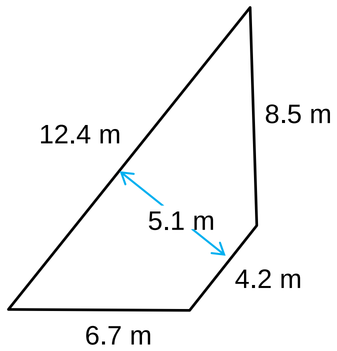
If the lengths of the parallel sides are a and b, then the average length is $\frac{a+b}{2}$.

Q32 Find the areas of these trapeziums:

(a)



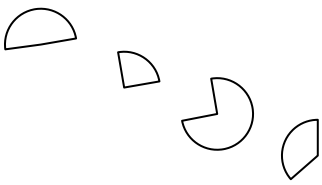
(b)



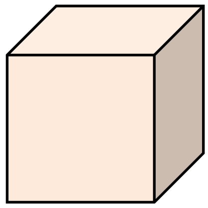
Areas of Sectors

Q33 A circle has a diameter of 8 cm. Find the areas of sectors of the circle which are:

(a) ½ of the circle (semicircle)
(b) ¼ of the circle (quadrant)
(c) ¾ of the circle
(d) $^1/_3$ of the circle
(e) $^2/_3$ of the circle
(f) $^3/_{10}$ of the circle
(g) $^1/_{360}$ of the circle
(h) $^{71}/_{360}$ of the circle
(i) A 71°sector
(j) A 225°sector



Surface Areas



A large MAB block is a 10 cm cube. It is 10 cm long, 10 cm wide and 10 cm high.

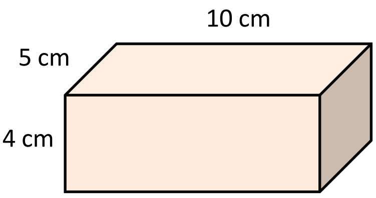
Q34 What is the area of the top face?

Q35 What is the combined area of all six faces added together?

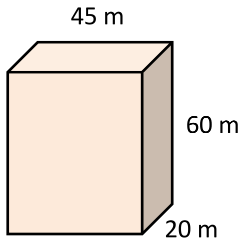
The combined area of all faces of a 3D shape is called its ***surface area***.

Q36 Find the surface areas of these shapes. The first four are prisms; the last is a symmetrical square-based pyramid. Don't forget the bottoms.

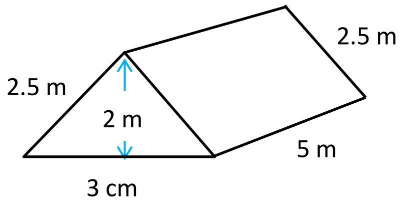
(a)



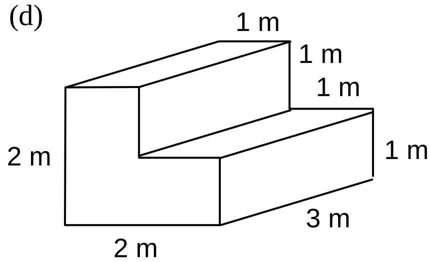
(b)



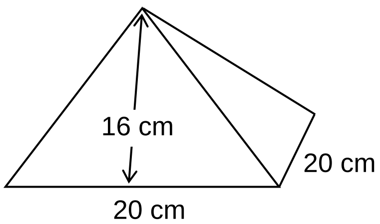
(c)



(d)



(e)

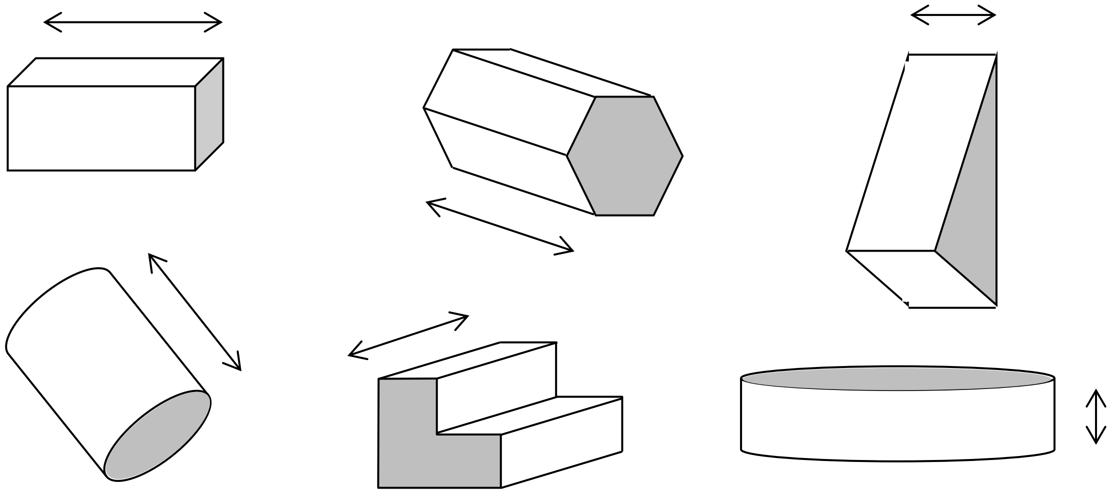


Volumes of Prisms and Cylinders

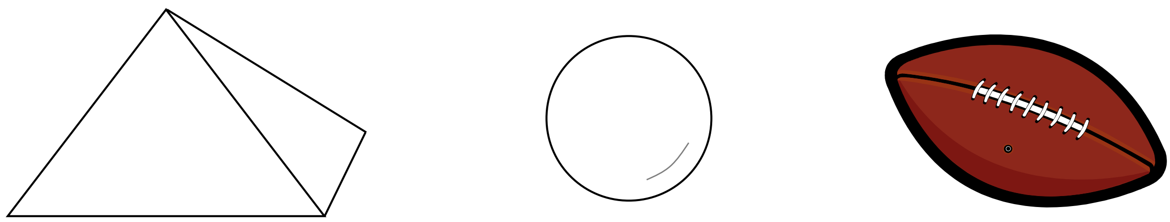
In Module M1-4 you learnt how to find the volumes of rectangular prisms. You should know that $V = l \times w \times h$, where V is the volume, l is the length, w is the width and h is the height. Go back and revise if you need to.

A prism has an end and a length. The end can be any size and shape. But any cut (cross-section) parallel to the end must be the same size and shape as the end. In other words, a prism is any 3D shape which stays the same shape and size as you move along its length. It can be sliced parallel to the end and all the slices will be the same. The length is the total thickness of all the slices.

The picture below shows a few prisms. The length of each is marked by the arrow and the end is shaded grey. Of course, the face opposite the grey one could also be the end.



The shapes below are not prisms. There is no way to slice them such that all the slices will be the same.



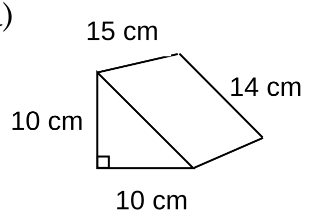
We could say that **a prism is a shape that can be cut into identical slices**.

Prisms can be named according to the shape of their end (or cross-section or slices). So the prisms above are: a rectangular prism, a hexagonal prism, a triangular prism, a circular prism (also known as a cylinder), a step-shaped prism and another cylinder. Cylinders like the last one, with length shorter than their width, are sometimes called discs (or disks in American and computer usage).

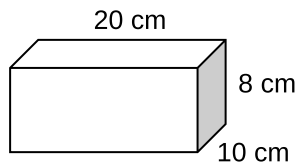
In the case of a rectangular prism, it doesn't matter which dimension is the length and which face is the end. But with other prisms, only one direction can be the length and only one of two opposite faces can be the end.

Q37 Use a 3D version of the envelope method with a rectangular prism as the envelope to find the volumes of these prisms.

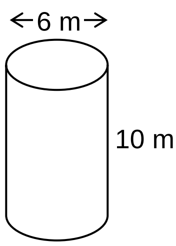
(a)



(b)



(c)



The 3D envelope method is basically $V = l \times w \times h \times f$.

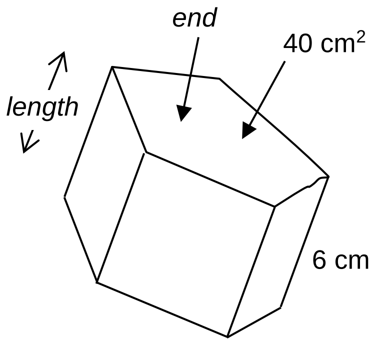
So for shape (a) $f = ½$ and $V = 10 \times 10 \times 15 \times ½$,

Q38 Write the same thing for shapes (b) and (c)

Note that $V = l \times w \times h \times f$ is the same as $V = w \times h \times f \times l$

$w \times h \times f$ is the area of the end.

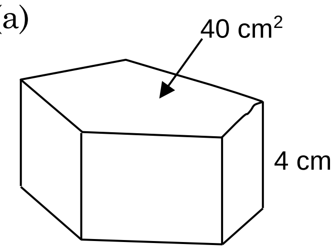
So we can say that the volume of a prism is equal to the area of the end multiplied by the length. ***V* = *A* × *l***.



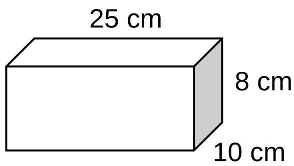
For this prism, A is 40, l is 6, so $V = 40 \times 6 = 240$ cm³.

Q39 Calculate the volumes of these shapes.

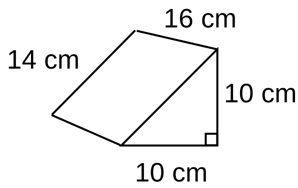
(a)



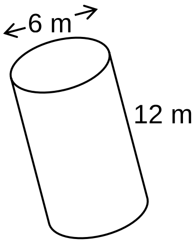
(b)



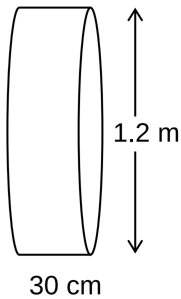
(c)



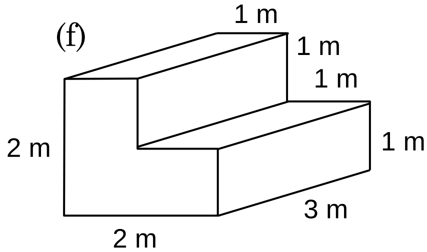
(d)



(e)

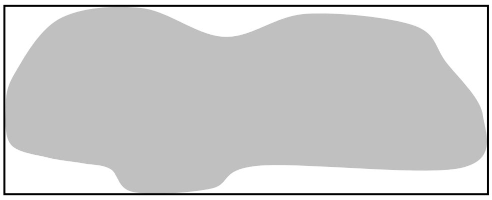


(f)



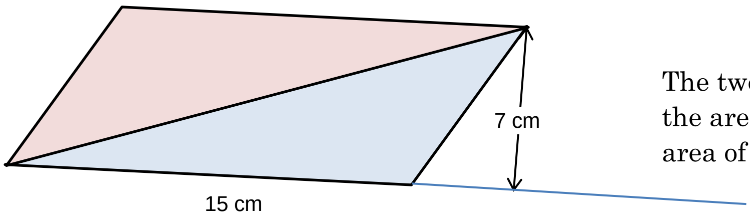
Answers

Q1 10 cm
Q2 20 cm
Q3 something like 30 cm
Q4 no
Q5 (a) 31.4 cm (b) 9.42 m
Q6 6.28 cm
Q7 7.07 cm
Q8 31.4 m
Q9 40.9 cm
Q10



Q11 depends on the size of your drawing

Q12 around 80 to 85% or 0.8 to 0.85
Q13 multiply your area by your fill factor
Q14 answers depend on your drawing
Q15 answers depend on your drawings
Q16 the shaded area in each smaller rectangle is equal to the unshaded area, so the total shaded area Is equal to the total unshaded area, so the triangle is half the area of the rectangle.
Q17 270 cm^2
Q18 (a) 6.5 cm^2 (b) 500 m^2 (c) 600 cm^2 (d) 20 km^2 (e) 20 cm^2
Q19 (a) 90 cm^2 (b) 300 cm^2 (c) 54 cm^2
Q20 around 75 to 80 cm^2
Q21 78.5 cm^2
Q22 (a) 12.57 cm^2 (b) 113 cm^2 (c) 0.080 m^2
Q23 10 cm^2
Q24 the length and the width measured perpendicular to the length
Q25 72 cm2
Q26 12×6, 8×9
Q27 126 cm2
Q28



The two triangles have the same area, so the area of the blue triangle is half the area of the parallelogram.

Q29 A: 3500 m^2, B: 1200 m^2 C: 2300 m^2 (C is what's left of the parallelogram.)
Q30 same
Q31 (a) 16 (b) 16 (c) 16 (d) 16
Q32 (a) 150 m^2 (b) 42.33 m^2
Q33 (a) 25.13 cm^2 (b) 12.57 cm^2 (c) 37.70 cm^2 (d) 16.76 cm^2 (e) 33.51 cm^2
(f) 15.08 cm^2 (g) 0.1396 cm^2 (h) 9.91 cm^2 (i) 9.91 cm^2 (j) 31.4 cm^2
Q34 100 cm2
Q35 600 cm2
Q36 (a) 220 cm^2 (b) 9600 m^2 (c) 46 m^2 (d) 30 m^2 (e) 1040 cm^2
Q37 (a) 750 cm^3 (b) 1600 cm^3 (c) 283 m^3
Q38 (b) $f = 1$ and $V = 20 \times 10 \times 8 \times 1$ (c) $f = {}^{\pi}/_{4}$ and $V = 10 \times 6 \times 6 \times {}^{\pi}/_{4}$
Q39 (a) 160 cm^3 (b) 2000 cm^3 (c) 800 m^3 (d) 339 m^3 (e) 0.339 m^3 or 3390 cm^3 (f) 9 m^3