M1 Maths

G4-1 Geometric Proofs

- proving geometric statements using chains of reasoning
- circle theorems

Summary   Lead In   Learn   Solve   Revise   Answers

Summary

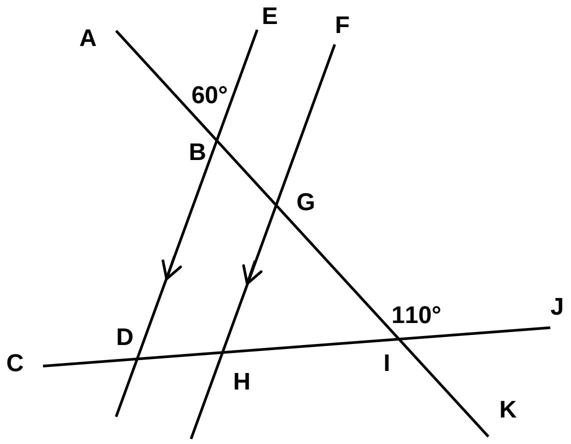
There is a standard way of recording the reasoning used to draw geometric conclusions using theorems.

Learn

Recording chains of reasoning / Proof

Sometimes we need not just to find an unknown angle on a diagram, but to show how we found it. We do this with a proof consisting of a chain of reasoning.

For example, suppose we need to find ∠ IHG in this diagram.



The answer is 50°. This is how we would prove it:

∠ ABE = 60° (given)

∴ ∠DBG = 60° (X-angles)

∴ ∠HGI = 60° (H-angles)

∠GIJ = 110° (given)

∴ ∠GIH = 70° (T-angles)

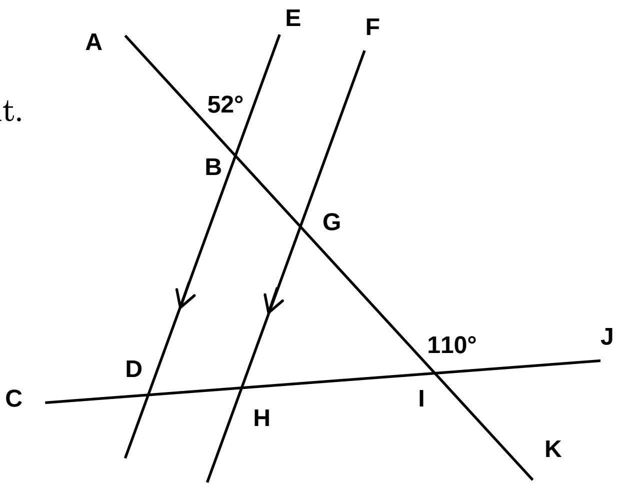
∴ ∠IHG = 50° (D-angles)

Note that we start from the angles we are given and deduce other angles from those, then further angles from those, and so on. The ∴ symbol means ‘therefore’. We put the theorem we used in brackets after each step. The last step will give us the angle we want.

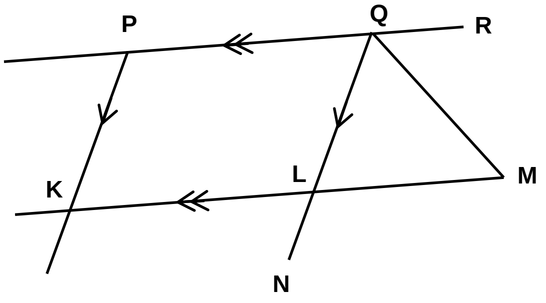
Sometimes, instead of being asked to find an angle, you may be asked to show that an angle has a certain size. For instance, in the example above, you might have been asked to show that ∠IHG = 50°. You working would be exactly the same.

Practice

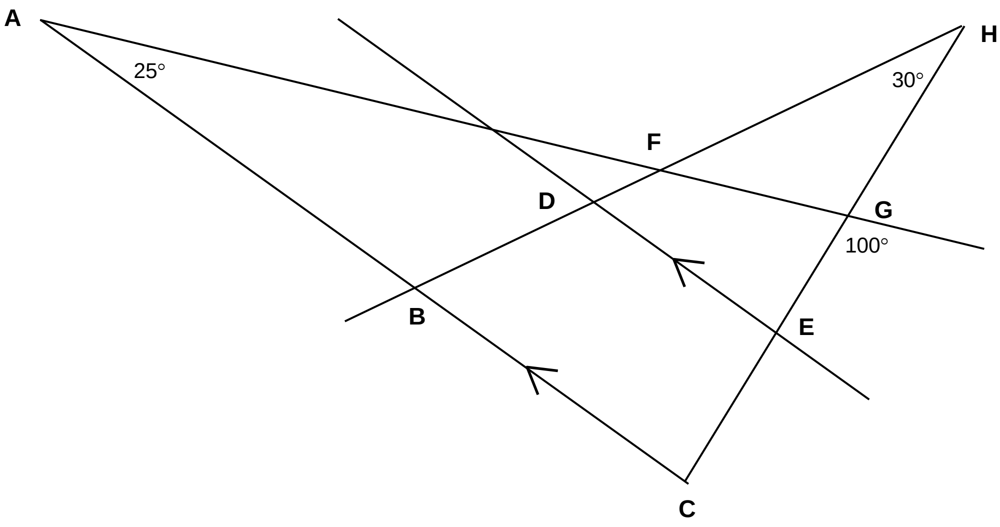
Q1 Find ∠IHG in the diagram to the right. Record your chain of reasoning.



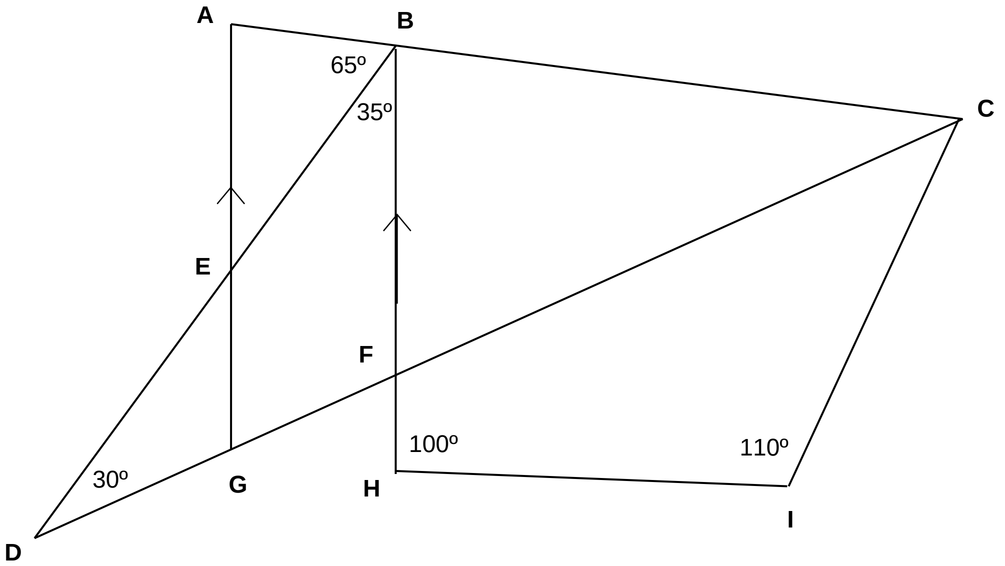
Q2 Given that ∠LKP = 75° and that ∠QLM = ∠QML, prove that ∠PQM = 105° in the diagram below.



Q3 Prove that ∠BDE = 105°.



Q4 Prove that $\angle FCI = 35°$ and that $\angle GEB = 145°$.



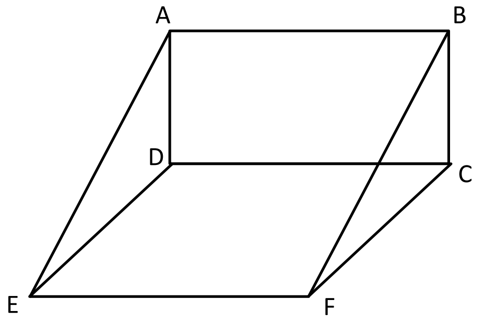
Geometric Proofs using Congruent Triangles

If two triangles are congruent, we can indicate this by writing $\Delta ABC \cong \Delta DEF$. We use the Δ symbol to mean 'triangle' and the $\cong$ symbol to mean 'is congruent to'. Note that the order of the letters matters. If the two triangles ΔABC and ΔDEF were to be exactly superimposed, then vertex A would have to be superimposed on vertex D, vertex B on Vertex E and vertex C on vertex F. Then we know that $\angle A = \angle D$, $\overline{BC} = \overline{EF}$ etc.

The main application of congruent triangles is in deducing angles and lengths on geometric figures and proving such results. If we can show that two triangles are congruent, then any side or angle on one will be equal to the corresponding side or angle on the other.

We will use the idea of congruence from Module G2-6 as well as other geometric ideas from G2-2 and G2-3. Go back and refresh your memory if necessary.

Example: ABCD is a rectangle and CDEF is a parallelogram. [Note this looks a bit like 3D figure, but it is 2D.]
Prove that $\overline{AE} = \overline{BF}$.



$\overline{AD} = \overline{BC}$ (opposite sides of a rectangle)

$\overline{DE} = \overline{CF}$ (opposite sides of a parallelogram)

$\angle BCD = \angle ADC = 90°$ (angles in a rectangle)

$\angle BCF = \angle BCD + \angle DCF$

$= 90° + \angle DCF$

$\angle CDE = 180° - \angle DCF$ (H-angles, opposite sides of a parallelogram are parallel)

$\angle$ADE = 360° – $\angle$ADC – $\angle$CDE (Y-angles)

= 360° – 90° – (180 – $\angle$DCF)

= 90° + $\angle$DCF

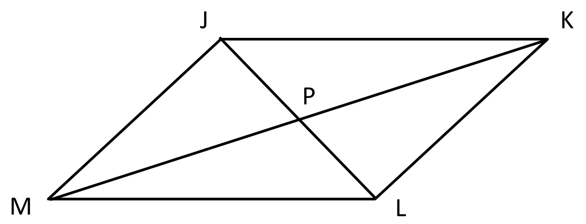
$\angle$ADE = $\angle$BCF

ΔADE $\cong$ ΔBCF (SAS)

$\therefore$ $\overline{AE} = \overline{BF}$

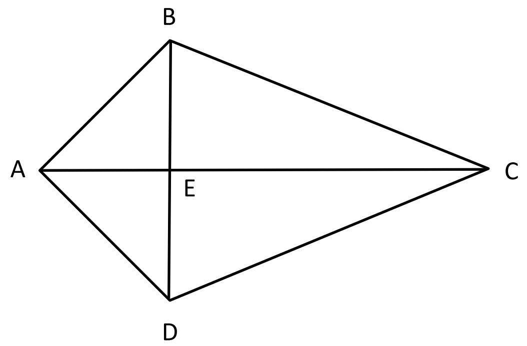
Practice

Q5 If JKLM is a parallelogram, prove that $\overline{MP} = \overline{PK}$.

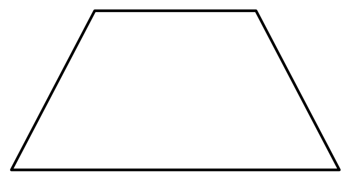


Q6 In this kite, $\overline{AB} = \overline{AD}$ and $\overline{CB} = \overline{CD}$. Prove that $\overline{BE} = \overline{ED}$.

Also prove that $\overline{BD}$ is perpendicular to $\overline{AC}$.



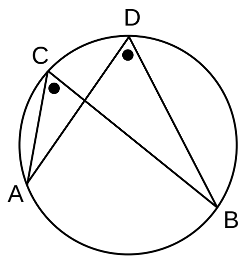
Q7 In this trapezium, the left and right sides are equal. Prove that the angles at the base are also equal. HINT: You may need to construct some extra lines so that you have congruent triangles.

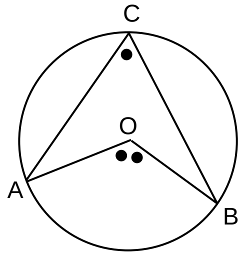


Circle Theorems

Circle theorems are used less frequently than TYXDOH or triangle theorems, but they aren't difficult. Four theorems are given here.

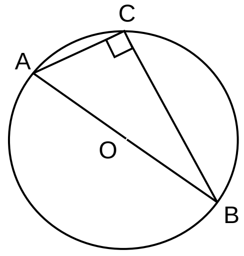
CM-angles: If A, B, C and D are four points on the circumference of a circle, and C and D are both on the same side of $\overline{AB}$, then $\angle$ACB = $\angle$ADB.

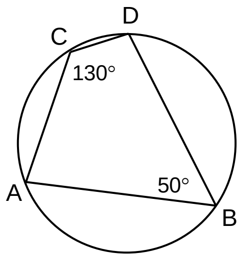




CA-angles: If A, B and C are three points on the circumference of a circle, and O is the centre of the circle, then ∠AOB = 2 × ∠ACB.

CD-angles: If A, B and C are three points on the circumference of a circle, and $\overline{AB}$ is a diameter, then ∠ACB = 90°.



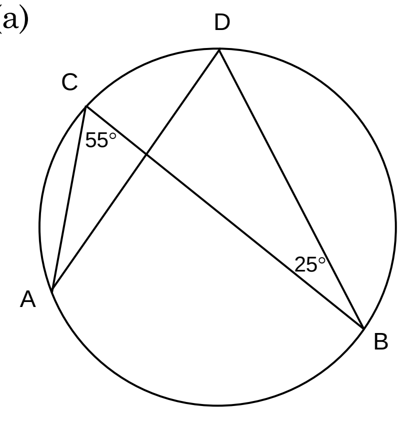


CO-angles: If A, B, C and D are four points on the circumference of a circle, then opposite angles of the quadrilateral ABCD add to 180°.

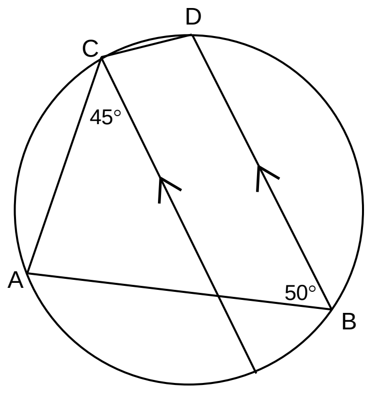
Practice

Q8 Copy these diagrams and write in all the angles.

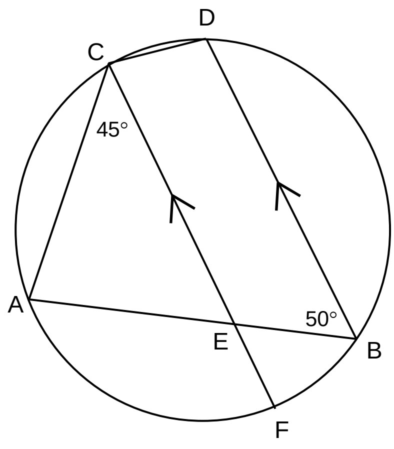
(a)



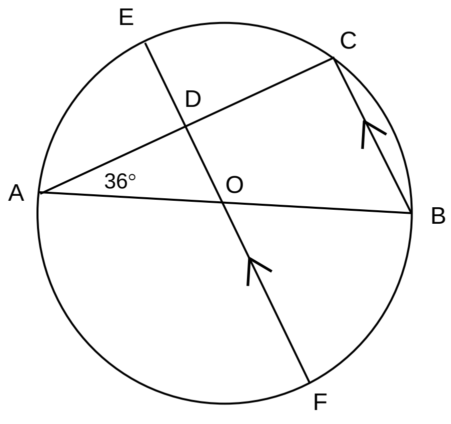
(b)



Q9 (a) Prove that ∠CDB = 95°



(b) If O is the centre of the circle and ∠CAB = 36°, prove that ∠AOF = 126°.



Solve

Q51 Use the T-angle theorem to prove the X-angle theorem.

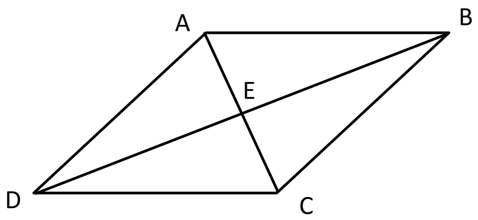
Note that you need to show that the theorem works for any angles, so you have to use variables for the angles rather than specific values. If you're not sure about this, have a look at how it's done in the Answers section, then try to reproduce it without copying.

Q52 Use circle theorem CA to prove circle theorem CD.

Q53 Using just the Z-angle theorem and the Y-angle theorem, prove the D-angle theorem.

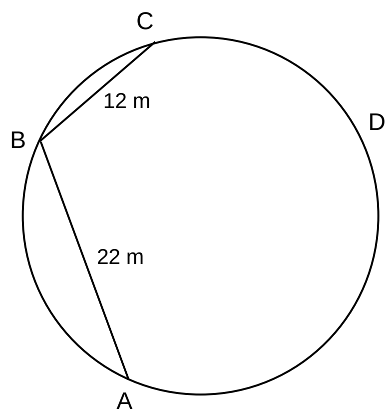
Q54 Using the D-angle theorem, show that the sum of the internal angles in any polygon with n sides is $180(n-2)°$.

Q55 Prove that the diagonals of this rhombus are perpendicular.



Q56 A shape will tessellate if a plane can be covered with multiple congruent copies of that shape without gaps or overlap. Show that triangles of any shape will tessellate.

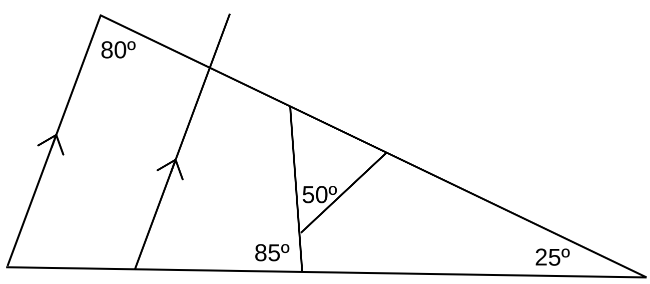
Q57 A circle had a diameter of 32 m. Persons A, B and C stand on the edge of the circle so that they are 22 m and 12 m apart as shown. Person D also stands on the circle to the right of C and A as shown. How far should D be from C in order to make ∠ADC as large as possible?



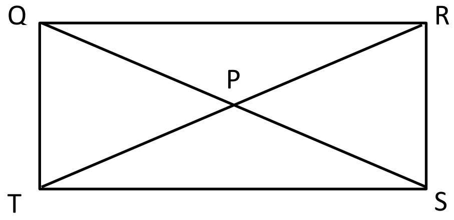
Revise

Revision Set 1

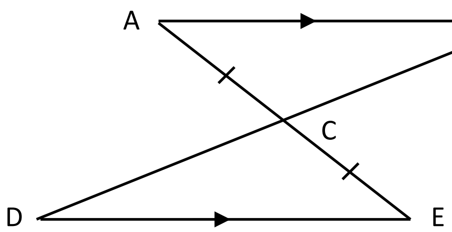
Q61 Copy this diagram and mark in the sizes of all the angles.



Q62 If QRST is a rectangle, prove that ∠PQT = ∠PTQ

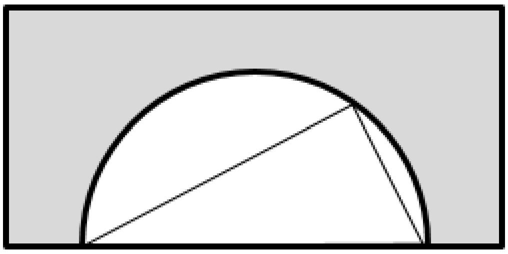


Q63 Prove that $\overline{BC} = \overline{CD}$



Q64 The space under a bridge has a semi-circular cross section with a diameter of 8 m. There is a hook on the roof. Wires are strung from the hook to the ground at either side of the bridge as shown. If the left wire makes an angle of 32° with the ground, what angle does the right wire make with the ground?

Explain your reasoning.



Answers

Q1 ∠ABE = 52° (given)
∴ ∠DBG = 52° (X-angles)
∴ ∠HGI = 52° (H-angles)
∠GIJ = 110° (given)
∴ ∠GIH = 70° (T-angles)
∴ ∠IHG = 58° (D-angles)

Q2 ∠LKP = 75° (given)
∴ ∠MLQ = 75° (H-angles)
∴ ∠NLK = 75° (X-angles)
∴ ∠LQP = 75° (H-angles)
∠LMQ = ∠MLQ (given) = 75°
∴ ∠MQL = 30° (D-angles)
∠PQM = ∠PQL + ∠LQM = 75° + 30° = 105°

Q3 ∠FGH = 100° (X-angles)
∴ ∠GFH = 50° (D-angles)
∴ ∠AFB = 50° (X-angles)

∴ ∠ABF = 105° (D-angles)

∴ ∠BDE = 105° (H-angles)

Q4 ∠BFD = 115° (D-angles)
∴ ∠HGC = 115° (D-angles)
∴ ∠FCI = 35° (O-angles)
∠AEB = 35° (H-angles)
∴ ∠GEB = 145° (T-angles)

Q5 As JKLM is a parallelogram, ∠KJP = ∠MLP (Z-angles)

∴ ∠JKP = ∠LMP (Z-angles)

As JKLM is a parallelogram, $\overline{JK} = \overline{LM}$

∴ KJP ≅ MLP (ASA)

∴ $\overline{PK} = \overline{MP}$

Q6 $\overline{AB} = \overline{AD}$ (given)

$\overline{CB} = \overline{CD}$ (given)

$\overline{AC}$ is common to ΔABC and ΔADC

∴ ΔABC ≅ ΔADC (SSS)

∴ ∠EAB = ∠EAD

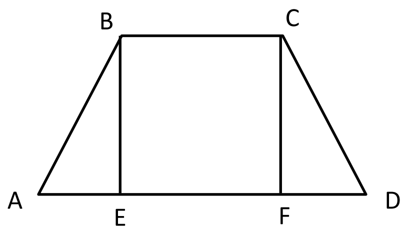
$\overline{AB} = \overline{AD}$ (given)

$\overline{AE}$ is common to ΔABE and ΔADE

∴ ΔABE ≅ ΔADE (SAS)

∴ $\overline{BE} = \overline{ED}$

Q7 Label the diagram and construct $\overline{BE}$ and $\overline{EF}$ perpendicular to $\overline{AB}$



As ABCD is a trapezium, $\overline{AB} \parallel \overline{EF}$

As ∠BEF = ∠CFD, $\overline{BE} \parallel \overline{CF}$

[NOTE: ∥ means ‘is parallel to’]

∴ BCFE is a parallelogram

As ∠BEF = 90°, BCFE is a rectangle

∴ $\overline{BE} = \overline{EF}$

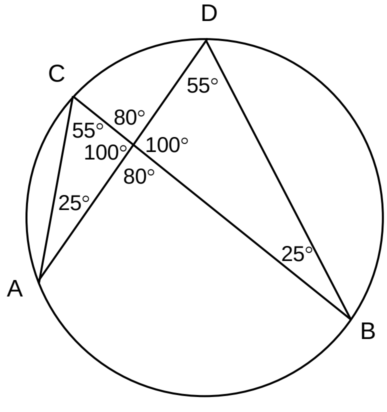
$\overline{AB} = \overline{CD}$ (given)

∠BEA = ∠CFD = 90°

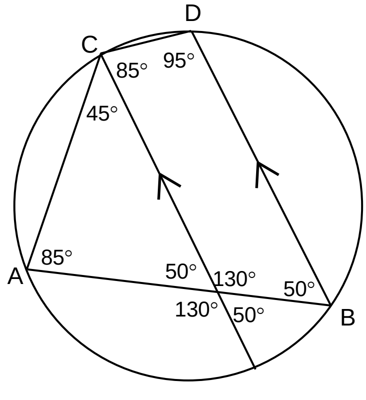
∴ ΔBEA ≅ ∠CFD

∴ ∠BAE = ∠CDF

Q8 (a)



(b)



Q9 (a) $\angle ABD = 50°$ (given)

$\therefore$ $\angle AEC = 50°$ (H-angles)

$\therefore$ $\angle BAC = 85°$ (D-angles)

$\therefore$ $\angle CDB = 95°$ (CO-angles)

(b) $\angle BAC = 36$ (given)

$\angle BCA = 90$ (CD-angles)

$\therefore$ $\angle ODA = 90$ (H-angles)

$\therefore$ $\angle AOD = 54$ (D-angles)

$\therefore$ $\angle AOF = 126$ (T-angles)

Q51 Let $\angle AEB$ be θ

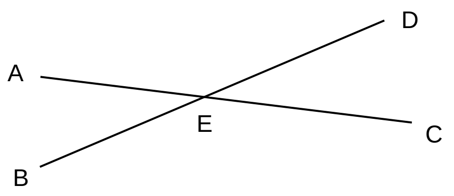
Then $\angle BEC = 180° - \theta$ (T-angles)

Then $\angle CED = 180° - (180° - \theta)$

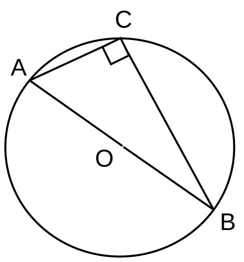
$= 180° - 180° + \theta$

$= \theta$

$\angle AEB$ always equals $\angle CED$



Q52



If AB is a diameter, then $\angle AOB = 180$

$\therefore$ $\angle ACB = 90$ (CA-angles)

Q53 Construct a line parallel to the base through the apex as shown.

Let $\angle BAD$ be a

Let $\angle ABD$ be b

$\angle ADC = a$ (Z-angles)

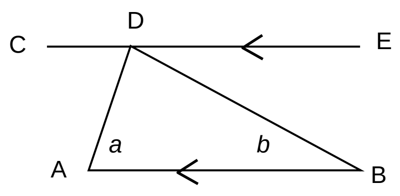
$\angle BDE = b$ (Z-angles)

$\angle CDE = 180°$

$\angle ADB = 360° - 180 - a - b$ (Y-angles)

$= 180° - a - b$

The three angles of the triangle add to $180° - a - b + a + b = 180°$



Q54 Let O be any point inside an n-sided polygon.

Connect O to each vertex of the polygon to make n triangles.

The sum of the angles in each triangle is 180.

$\therefore$ The total in the n triangles is $180n$.

360 of these are at O.

The rest form the internal angles of the polygon.

$\therefore$ The sum of the internal angles of the polygon is $180n - 360$.

$= 180n - 180 \times 2$

$= 180\,(n - 2)$

Q55 ABCD is a rhombus, so $\overline{AB} = \overline{BC} = \overline{CD} = \overline{DA}$

$\overline{DB}$ is common to ΔDBA and ΔDBC

$\therefore \Delta DBA \cong \Delta DBC$ (SSS)

$\therefore \angle DBA = \angle DBC$

$\overline{EB}$ is common to ΔEBA and ΔEBC

$\therefore \Delta EBA \cong \Delta EBC$ (SAS)

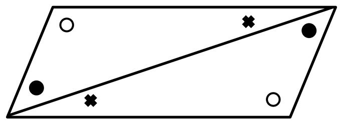
$\therefore \angle BEA = \angle BEC$

$\angle BEA + \angle BEC = 180°$

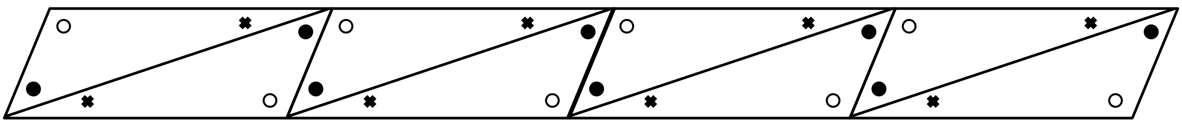
$\therefore \angle BEA = \angle BEC = 90°$

So the diagonals are perpendicular.

Q56 Any two congruent triangles can be put together by rotating one 180, then placing it next to the other with a pair of corresponding sides juxtaposed.



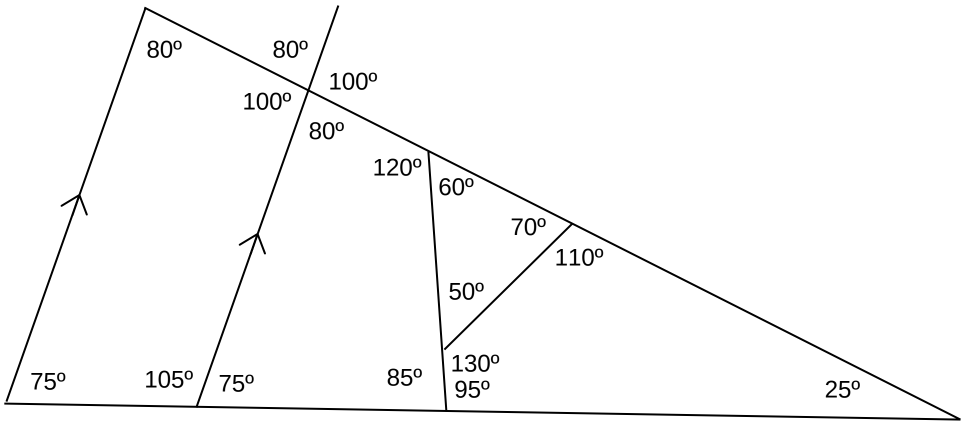
As opposite angles are then equal, the shape is a parallelogram.
Such parallelograms can be lined up to form a strip with straight parallel sides.



These strips can then be lined up side by side to cover the plane.

Q57 The angle will be the same whatever the distance. By the CO-angles theorem, it will always be $180° - \angle ABC$.

Q61



Q62 As QRST is a rectangle,

$\overline{QR} = \overline{TS}$

$\angle RQT = \angle STQ = 90$

$\overline{QT}$ is common to ΔRQT and ΔSTQ

$\therefore \Delta RQT \cong \Delta STQ$ (SAS)

$\therefore \angle PQT = \angle PTQ$

Q63 $\angle ABC = \angle CDE$ (Z-angles)

$\angle BAC = \angle CED$ (Z-angles)

$\angle ACB = \angle ECD$ (X-angles)

$\therefore \Delta ABC \cong \Delta EDC$ (AAS)

$\therefore \overline{BC} = \overline{CD}$

Q64 58°

By the CD theorem, the top angle of the triangle is 90°.

So the third angle must be 58°