M1 Maths

C6-8 Mastering Differentiation

- differentiating all functions fluently

Summary Lead-In Learn Solve Revise Answers

Summary

By the time you finish this module, you should be a master of differentiation, able to differentiate any compound function, however complex.

Learn

Introduction

By the time you finish this module, you should be able to differentiate any function you are likely to meet, however complex, and do it fluently. This is important if you are to be successful with the next modules on differential equations and integration. These involve anti-differentiation, which is just the reverse process of differentiation. It will be easy if you have thoroughly mastered differentiation; you will have trouble if you haven't.

The Chain Rule – Function of a Function

You probably learnt to use the chain rule using u and $\frac{du}{dx}$ etc. We can now use this approach on functions with exponential and trigonometric components, like this:

Let's differentiate $y = (e^x + \cos x)^4$

$$y = (e^x + \cos x)^4$$

$$y = u^4 \qquad u = e^x + \cos x$$

$$\frac{dy}{du} = 4u^3 \qquad \frac{du}{dx} = e^x - \sin x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 4u^3 \times (e^x - \sin x)$$

$$= 4(e^x + \cos x)^3 \times (e^x - \sin x)$$

Then you would have gone on to using the chain rule without writing this working, instead thinking about the *something*. In this case

$y = (e^x + \cos x)^4$

y = *something* to the power of 4

y' = the derivative of y with respect to the something multiplied by the derivative of the something with respect to x.

$= 4(e^x + \cos x)^3 \times (e^x - \sin x)$

Practice

Q1 Use the chain rule to differentiate the following. Don't forget to get the functions into differentiable form first by expressing fractions as negative powers and roots as fractional powers.

(a) $y = \sin x^3$ (b) $y = \sin (3x^2 + 5x)$ (c) $h = (\sin t + 1)^5$

(d) $y = (3x^2 + 5x)^8$ (e) $h = \dfrac{1}{(x^3 - 2x)^4}$ (f) $h = \sqrt{1 - x^2}$

(g) $y = (\ln x)^3$ (h) $h = \cos (2x)$ (i) $r = \ln (t - t^2)$

(j) $y = 2\cos (3t^2)$ (k) $P = e^{\sin x}$ (l) $s = e^{4x^2}$

(m) $y = \dfrac{1}{\sin x}$ (n) $t = \cos (e^x)$ (o) $A = 5^{-x^3}$

Q2 Use the chain rule to differentiate the following.

(a) $y = (2x^2 + x)^6$ (b) $y = (\sin x)^3$ (c) $h = (\cos r + r^2)^5$

(d) $y = \cos (4x^2 + 2x)$ (e) $h = \dfrac{1}{(\sin x)^3}$ (f) $h = \sqrt{4 - t^2}$

(g) $y = \sin (\ln x)$ (h) $h = \cos (5x^3)$ (i) $r = (t - t^2)^5$

(j) $y = 7 \sin (t^4)$ (k) $P = 2^{\sin x}$ (l) $s = e^{-2x}$

(m) $y = \dfrac{4}{\cos x}$ (n) $t = \sin (e^x)$ (o) $A = 3^{x^3}$

(p) $x = \ln (\sin t)$ (q) $p = \sqrt{\sin x + \cos x}$ (r) $y = \dfrac{1}{\sqrt{\ln x}}$

(s) $f = 3 \sin (1 - x^2)$ (t) $t = \sin s^2 - (\cos s)^2$ (u) $w = \dfrac{5}{(\cos t)^3}$

Some of the derivatives can be simplified and it is good style to simplify them where possible.

For example, Q9(s) gives $3\cos(1-x^2)\times -2x$. This can be simplified to $-6x\cos(1-x^2)$.

You should simplify answers where possible from here on and answers will be given in simplified form.

Practice

Q3 Differentiate the following, simplifying your answer where possible. Hint: Rewrite $\sin^2 x$ as $(\sin x)^2$ etc. before differentiating, but put it back into the $\sin^2 x$ form when simplifying.

(a) $y = \sin x^4$ (b) $y = \sin(2x^2+1)$ (c) $y = (\cos t + t)^3$

(d) $y = \ln 3x$ (e) $h = \cos(x+3)$ (f) $r = \ln(a + \sin a)$

(g) $y = -4\sin(3t^2)$ (h) $P = (7x^2 - 2x)^{11}$ (i) $s = 6e^{2x^3}$

(j) $y = \dfrac{7}{\cos x}$ (k) $t = 4x + 2\sin(e^x)$ (l) $A = 3 \times 2^{-x^3}$

(m) $x = \ln(t^2 + \cos t)$ (n) $p = \sqrt[3]{n+n^2}$ (o) $y = \dfrac{1}{4x+2}$

(p) $f = \sin(4x - x^2)$ (q) $t = \cos s^2 - \cos^2 s$ (r) $b = \dfrac{-2}{(\cos t)^7}$

The Chain Rule – Function of a Function of a Function

Suppose we need to differentiate $\sin e^{2x}$.

$2x$ is a function; e^{2x} is a function of a function; and $\sin e^{2x}$ is a function of a function of a function.

Differentiating a function of a function of a function is really no harder than differentiating a function of a function.

Again we look at the outside function, in this case *sine of something*.

The derivative is *cos of something*, in this case $\cos e^{2x}$.

Then, because the *something* isn't just x, we have to multiply by the derivative of the *something*.

Now, in this case, the *something* is a function of a function and so requires the chain rule. Its derivative is $e^{2x} \times 2$.

So the whole derivative is $\cos e^{2x} \times e^{2x} \times 2$, which should be simplified to $2e^{2x}\cos e^{2x}$.

A function of a function of a function of a function can be differentiated in the same way. Just start with the outside function. Then the derivative of the something will be the derivative of a function of a function of a function.

You might be able to see why this rule is called *the chain rule*.

Practice

Q4 Differentiate the following. Don't forget to simplify where appropriate.

(a) $y = \cos e^{4x}$ (b) $y = \sin e^{x^5}$ (c) $y = \ln (\cos t)^2$

(d) $y = \sin (\ln 6x)$ (e) $h = e^{\cos^2 x}$ (f) $r = \ln (\sin a^2)$

(g) $w = -12\sin (\ln x)^2$ (h) $P = (\cos 4x - 2x)^{11}$ (i) $h = 3e^{4\cos^3 x} - e^{sin^2 x}$

(j) $r = 2 \sin (e^{\sqrt{\ln x}})$ (k) $y = \ln \left(\cos \left(\frac{2}{\sqrt{\sin(x+1)}}\right)\right)$

You should get to the point of being able to use the chain rule fairly automatically without having to puzzle over it or to remind yourself of what to do. Keep practising until you get to that point.

The Product Rule

In this section you will use the product rule in conjunction with sin x, cos x, a^x e^x and ln x. Remember that the product rule is used to differentiate the product of two factors. The derivative is the product of the factors with one of them differentiated plus the product of the factors with the other differentiated.

So to differentiate $y = 4x \sin x$, we write

$y = 4x \sin x$

$y' = 4 \sin x + 4x \cos x$

Practice

Q5 Use the product rule to differentiate the following, laying out the working as shown above.

(a) $x = x^3 \sin x$ (b) $y = e^x \sin x$ (c) $y = (x^2 + 5x) \cos x$

(d) $y = x \ln x$ (e) $h = \sin x \cos x$ (f) $r = 2^x \ln x$

(g) $y = -4e^t \sin t$ (h) $P = (3x^2 - 5x)(4 - x)$ (i) $s = 6e^x (x^2 - 2)$

(j) $k = \sqrt{x} \ln x$ (k) $t = -4x \cos x$ (l) $A = 3x \times 2^x$

Q6 Use the product rule to differentiate the following without writing any working.

(a) $y = x^2 \sin x$ (b) $v = (5x^3 + 1)\, e^x$ (c) $P = (\cos t + t) \sin t$

(d) $y = e^x \sin x$ (e) $h = (x^2 - 5x) \ln x$ (f) $r = (\cos w + 2 \sin w) \sin w$

(g) $y = 1.05^x \times x^2$ (h) $P = (7x^2 - 2x) \cos x$ (i) $s = 4\,(x^3 - x)\, e^x$

(j) $f = 2x^2 e^x$ (k) $t = 4x \sin x$ (l) $A = \ + \sin x$

(m) $x = t \ln t - e^t$ (n) $p = \sqrt[3]{n^2} \cos n$ (o) $y = 5 \sin x \cos x$

(p) $f = 3\, e^x \sin x$ (q) $t = \cos s - s \cos s$ (r) $b = (3r^2 - 4r)(\cos r - \sin r)$

If you meet an expression which is a product of 3 functions, the product rule can be adapted to $y' = u'vw + uv'w + uvw'$. This time we write out uvw 3 times, first time with the u differentiated, second time with the v differentiated and third time with the w differentiated.

You can probably extrapolate to the formula for a product of 4 functions, though you might run out of letters.

Practice

Q7 Use the product rule to differentiate the following.

(a) $y = x^2 e^x \sin x$ (b) $y = 2^x (x^3 + 2x^2) \cos x$ (c) $y = (x^2 + 5x)\, e^x \ln x$

The Quotient Rule

The quotient rule can be used for any function which can be written as one expression divided by another expression.

Consider $y = \dfrac{e^x}{x^2 + 2x}$

We let the top expression (the numerator) be u and the bottom expression (the denominator) be v. Again, we let their derivatives be u' and v'.

We write:

$$y = \frac{e^x}{x^2 + 2x}$$

$u = e^x$ $v = x^2 + 2x$

$u' = e^x$ $v' = 2x + 2$

$$y' = \frac{u'v - uv'}{v^2}$$

$$= \frac{e^x(x^2 + 2x) - e^x(2x + 2)}{(x^2 + 2x)^2}$$

$$= \frac{(x^2 - 2)e^x}{(x^2 + 2x)^2}$$

As with the product and chain rules, you might be comfortable doing this without the working, though as the quotient rule formula is more complex and less symmetrical, it can be worth continuing to use it.

Practice

Q8 Use the quotient rule to differentiate the following.

(a) $y = \frac{x^2}{x+3}$ (b) $y = \frac{x^2 - x}{x^2 + x}$ (c) $y = \frac{\sin x}{x}$

(d) $y = \frac{e^x}{x^2}$ (e) $y = \frac{2^x}{\cos x}$ (f) $y = \frac{\ln x}{x+1}$

(g) $y = \frac{\sin x}{\cos x}$ (h) $y = \frac{\cos x}{\sin x}$ (i) $y = \tan x$

(j) $y = \frac{4x}{\ln x}$ (k) $y = \frac{\ln 2x}{\ln x}$ (l) $y = \frac{e^x}{\ln x}$

(m) $y = \frac{4x^2 - 6x}{x^2 + 3}$ (n) $y = \frac{x^2}{x+3}$ (o) $y = \frac{2 \ln x}{1 + \sin x}$

Mixed Practice

The questions in the next exercise require the chain rule, product rule or quotient rule. They will give you practice in choosing as well as more practice in using the rules.

Practice

Q9 Use the chain rule, product rule or quotient rule to differentiate the following.

(a) $y = \sin x^4$ (b) $y = x^2 \sin x$ (c) $s = 6e^{2x^3}$

(d) $y = \frac{\ln x}{x^3}$ (e) $h = e^x \cos x$ (f) $r = \ln(a + \sin a)$

(g) $y = -4\sin(3t^2)$ (h) $P = x^9 + (7x^2 - 2x)^{11}$ (i) $y = \frac{1}{(t + \sin t)}$

(j) $y = \frac{7x}{\cos x}$ (k) $t = 4x + 2\sin(e^x)$ (l) $A = 8c\sin c$

(m) $x = \ln(t^2 + \cos t)$ (n) $p = \sqrt[3]{n + n^2}$ (o) $y = \frac{1}{4x+2}$

(p) $f = xe^x$ (q) $t = \sin s^2 + \sin^2 s$ (r) $b = \frac{-2e^t}{\cos t}$

Q10 Use the chain rule, product rule or quotient rule to differentiate the following.

(a) $y = \cos(x^3 + e^x)$ (b) $z = \sin 6x$ (c) $m = 4x\ln x$

(d) $y = \frac{\sin x}{x}$ (e) $h = -2e^x$ (f) $r = 4e^{5s}$

(g) $y = -4t^6\sin t$ (h) $P = \frac{x^4}{3x^5 - x^2}$ (i) $y = \frac{1}{(n - \ln n)}$

(j) $y = \frac{7x^2}{\cos x}$ (k) $t = 4e^{3x} + 2\sin 3x$ (l) $A = \ln c\sin c$

(m) $x = e^{3t - \sin t}$ (n) $p = \sqrt[4]{\sin r}$ (o) $y = \frac{5}{3x^2 + 2x}$

(p) $f = x^2e^x$ (q) $a = \cos x^2 + 3\cos^2 x$ (r) $x = \frac{2^t}{\sin t}$

Rules in Combination

Some functions require more than one application of the chain, product and quotient rules.

$y = x^2\sin 3x$ is an example. $y = (\ln(x^2 + 2x))^3$ is another.

To differentiate these, we look at the overall structure: if a vertical line can separate it into two functions multiplied together, then start with the product rule; if it is a function over another function, then start with the quotient rule; otherwise, if it is a function of a function, start with the chain rule

Example 1

Taking $y = x^2\sin 3x$, this is a product, so we think $u'v + uv'$

and write $2x\sin 3x + x^2 \times$ . . . .

But, of course, differentiating $\sin 3x$ requires the chain rule, giving us $\cos 3x \times 3$.

So we get $2x\sin 3x + x^2 \times 3\cos 3x$, which can be simplified to $2x\sin 3x + 3x^2\cos 3x$.

Example 2

Taking $y = (\sin(x^2 + 2x))^3$, this is a function of a function, so we use the chain rule to get

$3(\sin(x^2 + 2x))^2 \times$ the derivative of $\sin(x^2 + 2x)$.

But differentiating $\sin(x^2 + 2x)$ requires us to use the chain rule again to get $\cos(x^2 + 2x) \times (2x + 2)$.

So we get $3(\sin(x^2 + 2x))^2 \times \cos(x^2 + 2x) \times (2x + 2)$,
which can be simplified to $(6x + 6)\sin^2(x^2 + 2x)\cos(x^2 + 2x)$.

Basically, you just follow your nose and it should come out right. The following questions will give you the practice you need.

Practice

Q11 Use the chain rule, product rule and/or quotient rule to differentiate the following.

(a) $y = x \sin 2x$ (b) $y = \ln(x^2 \sin x)$ (c) $s = 6x^2 e^{2x^3}$

(d) $y = \frac{\ln x^2}{x^3}$ (e) $h = e^{1+x} \sin 2x$ (f) $r = \frac{e^{3x}}{\sin 3x}$

(g) $y = \sin(3t^2 e^t)$ (h) $P = (\sin 2x - 2x)^5$ (i) $y = \frac{e^{2t}}{(t - \cos t)}$

(j) $y = \frac{x}{\sin x^2}$ (k) $t = 4x e^x \sin x$ (l) $A = 3b \sin(6b^2 + 2b)$

(m) $x = \ln(t^4 \sin t^2)$ (n) $p = \sqrt{n + \sin n^2}$ (o) $y = \sqrt{\frac{e^x}{\cos x}}$

(p) $f = 2x + 3 - x^4 e^{6x}$ (q) $t = [\ln(\sin 2s)]^3$ (r) $b = \frac{-2e^{5t}}{\cos t^2}$

Q12 Differentiate the following.

(a) $p = x^3 \sin 5x$ (b) $b = \frac{\cos x}{2^x}$ (c) $y = (\cos t + 3t)^3$

(d) $y = x \ln 3x$ (e) $h = 2x^2 \cos(x^2 + 3)$ (f) $r = \ln(a + \sin a^2)$

(g) $y = -4t^2 \sin(3t^2)$ (h) $P = \cos^2(7x^2 - 2x)$ (i) $s = \ln(x \cos x)$

(j) $y = \frac{x^2}{\cos x}$ (k) $h = \sin x + \tan x$ (l) $A = 3x \times 2^{-x^3}$

(m) $z = \frac{t^2}{4t^3 - 2t}$ (n) $g = \sqrt{x \cos x}$ (o) $x = 4 \ln(t^2 \cos t)$

(p) $p = n \times \sqrt[3]{n + n^2}$ (q) $y = \dfrac{\sin 2x}{4x + 2}$ (r) $t = \cos s^2 \sin^2 s$

(s) $c = \dfrac{-2b}{(\cos b)^5}$ (t) $f = \cos x \sin (4x - x^2)$ (u) $y = \cos \theta \tan \theta$

And a few word problems:

Practice

Q13 The tide height in metres at Wilhelm Bay last Saturday was given by $h = 1.8 \cos \frac{1}{2}(t - 3) + 2.2$, where t is the time in hours since midnight. How fast was the tide rising or falling at 10 am?

Q14 The population of Nobbsville is given by $p = 2212e^{0.0035t}$, where t is the number of years since 1990. How fast was the population growing in 2017?

Q15 The concentration, c, in mg/L of menthoxinol in the blood t hours after taking a dose is given by $c = 3te^{-t}$.

(a) Find the rate at which the concentration is changing 4 hours after taking a dose.

(b) Find the time at which the concentration is at a maximum.

(c) Find the maximum concentration.

Q16 Julie's height in centimetres between the ages of 4 and 12 was given by $h = 65 \times 1.1^a$, where a is her age in years. Her mass in kilograms is given by $m = 0.004a \times h^2$. At what rate was her mass increasing on her 9th birthday.

Q17 The average price of houses in Blurbledale in thousands of dollars between 1994 and 2008 was given by $p = \frac{t^2 - 42t + 1320}{5 + t}$, where t is the number of years since 1994. How fast were prices rising or falling in 2002?

Solve

Q51 Differentiate $\ln x \times \sin (x^2 e^{-\cos 3x})$.

Q52 Differentiating $y = x^x$ is a bit of a challenge. It can be done with what you know about differentiation and some manipulation using what you know about logs. Try it.

Q53 A pendulum swings such that its displacement, x, to the right of its equilibrium position is given by $x = 8 e^{-0.04t} \cos 2t$, where x is in metres and t is the time in seconds. The exponential factor is there because the amplitude of the swings decreases exponentially with time. Find an expression for the horizontal velocity and hence find the velocity the fourth time it is at the equilibrium position.

Revise

Revision Set 1

Q61 The number of Daleks that have landed in the English countryside is increasing by 20% per day. If there were 350 at 4 pm on April 5. At what rate were they arriving at 4 am on April 16?

Q62 Differentiate:

(a) $y = (x^2 + 7x)^8$ (b) $y = x^2 e^x$ (c) $y = \frac{2x}{x^2+5}$

(d) $p = r^2 \ln r$ (e) $m = \frac{\sin x}{\cos x}$ (f) $y = \sin (x^2 + 2e^x)$

(g) $y = \cos (x^3 e^x) + x$ (h) $y = (x^3 + 7x) \sin^2 3x$

Q63 Arthur walks 3 km east from camp, then 4 km north. He is then 5 km from camp. After that, he continues to walk north at 1 km/h. At what rate is his distance from camp increasing after he has walked a further 2 km, i.e. when he is 7 km further north than camp.

Revision Set 2

Q71 A Ferris wheel is turning such that the height of Car No 1 is given by $h = 8 \sin t + 3$, where h is the height in metres and t the time since it started in minutes. What was the vertical component of Car No 1's velocity 7 minutes after starting?

Q72 Differentiate:

(a) $y =$ $x^2 \cos x$ (b) $y = \tan \theta$ (c) $h = 4e^{1-0.1t} + 3t$

(d) $s = \frac{\sqrt{x}}{\sin x}$ (e) $y = \sin (3 \ln x)$ (f) $y = 2 \cos x \ln x$

(g) $y = \ln (x^2 \sin 5x)$ (h) $y = \frac{\sin(x^2-3x)}{2^{4x}}$

Q73 A weight hanging on the end of a string is swinging, but the amplitude of the swings is slowly decreasing. It's sideways displacement t seconds after starting is given by $x = e^{-0.02t} \cos 2t$. What is the horizontal component of its velocity when $t = 12$?

Revision Set 3

Q81 At what value of x is the graph of $y = \ln x - x$ horizontal?

Q82 Differentiate:

(a) $y = 7 \times 10^x \times \cos x$ (b) $A = \frac{e^x}{x^2} + \sin x$ (c) $y = 2\sqrt[3]{\sin x}$

(d) $y = \frac{4x^7}{4x+2}$ (e) $h = 4 \ln \cos t$ (f) $k = e^x \cos x$

(g) $y = 3 \sin x \cos^2 (3x + 4)$ (h) $b = \ln\left(\frac{-2t}{\cos t}\right)$

Q83 If t is the time in days since the start of April, the probability that it will have rained by time t is given by $p = \frac{1}{4}\ln (t + 1) \div \cos 0.005t$. At what rate is the probability increasing when $t = 20$?

Answers

Q1 (a) $s' = 3x^2 \cos x^3$ (b) $y' = (6x + 5)\cos(3x^2 + 5x)$ (c) $y' = 5 \cos t(\sin t + 1)^4$

(d) $y' = 8(3x^2 + 5x)^7 (6x+5)$ (e) $h' = \frac{-4(3x^2-2)}{(x^3-2x)^5}$ (f) $h' = \frac{x}{\sqrt{1-x^2}}$

(g) $y' = \frac{3(\ln x)^2}{x}$ (h) $h' = -2\sin 2x$ (i) $r' = \frac{1-t}{t-t^2}$

(j) $y' = -12 \sin (3t^2)$ (k) $P' = \cos x\, e^{\sin x}$ (l) $s' = 8xe^{4x^2}$

(m) $y' = \frac{-\cos x}{\sin^2 x}$ (n) $t' = -e^x \sin (e^x)$ (o) $A' = \frac{-3x^2 \ln 5}{5^{x^3}}$

Q2 (a) $y' = 6(2x^2 + x)^5(4x + 1)$ (b) $y' = 3(\sin x)^2 \cos x$ (c) $h' = 5(\cos r + r^2)^4(2r - \sin r)$

(d) $y' = -(8x + 2) \sin (4x^2 + 2x)$ (e) $h' = \frac{-3\cos x}{(\sin x)^4}$ (f) $h' = \frac{-t}{\sqrt{4-t^2}}$

(g) $y' = \frac{\cos(\ln x)}{x}$ (h) $h' = -15x^2 \sin (5x^3)$ (i) $r' = 5(t - t^2)^4(1 - 2t)$

(j) $y' = 28\, t^3 \cos (t^4)$ (k) $P' = 2^{\sin x} \ln 2 \cos x$ (l) $s' = -2e^{-2x}$

(m) $y' = \frac{4\sin x}{\cos^2 x}$ (n) $t' = e^x \cos (e^x)$ (o) $A' = 3x^2 \times 3^{x^3} \ln 3$

(p) $x' = \frac{\cos t}{\sin t}$ (q) $p' = \frac{\cos x - \sin x}{2\sqrt{\sin x - \cos x}}$ (r) $y' = \frac{-1}{2x \ln x^{3/2}}$

(s) $f' = 6x \cos (1 - x^2)$ (t) $t' = 2s \cos s^2 - 2\cos s \sin x$ (u) $w' = \frac{25 \sin t}{(\cos t)^6}$

Q3 (a) $y' = 4x^3 \cos x^4$ (b) $y' = 4x \cos (2x^2 + 1)$ (c) $y' = 3(\cos t + t)^3(1 - \sin t)$

(d) $y' = \frac{1}{x}$ (e) $h' = -\sin (x + 3)$ (f) $r' = \frac{1+\cos a}{a+\sin a}$

(g) $y' = 24t \cos (3t^2)$ (h) $P' = (7x^2 - 2x)^{10}(154x - 22)$ (i) $s' = 36xe^{2x^3}$

(j) $y' = \frac{7\sin x}{\cos^2 x}$ (k) $t' = 4 + 2e^x \cos (e^x)$ (l) $A' = -9x^2 \times 2^{-x^3} \ln 2$

(m) $x' = \frac{2t - \sin t}{t^2 + \cos t}$ (n) $p' = \frac{1+2n}{3\sqrt[3]{n^4+2n^3+n^2}}$ (o) $y' = \frac{-4}{(4x+2)^2}$

(p) $f' = (4 - 2x)\cos(4x - x^2)$ (q) $t' = -2s \sin s^2 + 2 \sin s \cos^2 s$ (r) $b' = \frac{14 \sin t}{(\cos t)^8}$

Q4 (a) $y' = -4e^{4x} \sin e^{4x}$ (b) $y' = 5x^4 e^{x^5} \cos e^{x^5}$ (c) $y' = -2 \tan t$

(d) $y' = \frac{\cos(\ln 6x)}{x}$ (e) $h' = -2 \sin x \cos x\, e^{\cos^2 x}$ (f) $r' = \frac{2a}{\tan a^2}$

(g) $w' = \frac{-24 \ln x \cos(\ln x)^2}{x}$ (h) $P' = -11(4 \sin 4x - 2)(\cos 4x - 2x)^{10}$

(i) $s' = -36 \sin x \cos^2 x\, e^{4\cos^3 x} - 2 \sin x \cos x\, e^{\sin^2 x}$

(j) $r' = \frac{e^{\sqrt{\ln x}} \cos e^{\sqrt{\ln x}}}{x\sqrt{\ln x}}$ (k) $y' = \frac{2\cos(x+1)\ \tan\left(2/\sqrt{\sin(x+1)}\right)}{\sqrt{\sin^3(x+1)}}$

Q5 (a) $x' = 2x^2 \sin x + x^3 \cos x$ (b) $y' = e^x(\sin x + \cos x)$

(c) $y' = (2x + 5)\cos x - (x^2 + 5x)\sin x$

(d) $y' = \ln x + 1$ (e) $h' = \cos^2 x - \sin^2 x$ (f) $r' = 2^x \ln 2 \ln x + \frac{2^x}{x}$

(g) $y' = -4e^t(\sin t + \cos t)$ (h) $P' = 24x - 20 - 9x^2$ (i) $s' = (6x^2 + 12x - 12)e^x$

(j) $k' = \frac{\ln x + 2}{2\sqrt{x}}$ (k) $t' = 4x \sin x - 4 \cos x$ (l) $A' = 2^x(3 + 3x \ln 2)$

Q6 (a) $y' = x^2 \cos x + 2x \sin x$ (b) $v' = (5x^3 + 15x^2 + 1)e^x$

(c) $P' = \cos^2 t + t \cos t + \sin t - \sin^2 t$

(d) $y' = e^x(\sin x + \cos x)$ (e) $h' = (2x - 5)\ln x + x - 5$

(f) $r' = (2\cos w - \sin w)\sin w + (\cos w + 2\sin w)\cos w$

(g) $y' = 1.05^x \ln 1.05 \times x^2 + 1.05^x \times 2x$ (h) $P' = (14x - 2)\cos x + (7x^2 - 2x)\sin x$

(i) $s' = (4x^3 + 12x^2 - 4x - 4)e^x$

(j) $f = (2x^2 + 4x)e^x$ (k) $t' = 4(\sin x + x \cos x)$ (l) $A' = + \cos x$

(m) $x' = \ln t + 1 - e^t$ (n) $p' = \frac{2\cos n}{3\sqrt[3]{n}} - \sqrt[3]{n^2}\sin n$ (o) $y' = 5(\cos^2 x - \sin^2 x)$

(p) $f' = 3e^x \sin x + 3e^x \cos x$ (q) $t' = (s - 1)\sin s - \cos s$

(r) $b' = (6r - 4)(\cos r - \sin r) - (3r^2 - 4r)(\cos r + \sin r)$

Q7 (a) $y' = 2x\, e^x \sin x + x^2 e^x \sin x\ \ x^2 e^x \cos x$

(b) $y' = 2^x \ln 2\,(x^3 + 2x^2)\cos x + 2^x(3x^2 + 4x)\cos x - 2^x(x^3 + 2x^2)\sin x$

(c) $y' = (x^2 + 8x + 10)\, e^x \ln x$

Q8 (a) $y' = \frac{x^2+6x}{(x+3)^2}$ (b) $y' = \frac{(2x-1)(x^2+x)-(2x+1)(x^2-x)}{(x^2+x)^2}$ (c) $y' = \frac{x\cos x - \sin x}{x^2}$

(d) $y' = \frac{(x-2)e^x}{x^3}$ (e) $y' = \frac{2^x(\ln 2 \cos x + \sin x)}{\cos^2 x}$ (f) $y' = \frac{x+1- x\ln x}{x(x+1)^2}$

(g) $y' = \frac{1}{\cos^2 x}$ (h) $y' = \frac{-1}{\sin^2 x}$ (i) $y' = \frac{1}{\cos^2 x}$

(j) $y' = \frac{4(\ln x - 1)}{(\ln x)^2}$ (k) $y' = \frac{-\ln 2}{x(\ln x)^2}$ (l) $y' = \frac{(x\ln x - 1)e^x}{x(\ln x)^2}$

(m) $y' = \frac{(x^2+3)(8x-6) - 2x(4x^2 - 6x)}{(x^2+3)^2}$ (n) $y' = \frac{x^2+6x}{(x+3)^2}$

(o) $y' = \frac{2 + 2\sin x - 2x\ln x\cos x}{x(1+\sin x)^2}$

Q9 (a) $y' = 4x^3 \cos x^4$ (b) $y' = x^2 \cos x + 2x \sin x$ (c) $s' = 36x^2 e^{2x^3}$

(d) $y' = \frac{1-3\ln x}{x^4}$ (e) $h' = e^x (\cos x - \sin x)$ (f) $r' = \frac{1+\cos a}{a+\sin a}$

(g) $y' = -24t\cos(3t^2)$ (h) $P' = 9x^8 + 11(7x^2 - 2x)^{10}(14x - 2)$ (i) $y' = \frac{-1-\cos t}{(t+\sin t)^2}$

(j) $y' = \frac{7\cos x + 7x\sin x}{\cos^2 x}$ (k) $t' = 4 + 2e^x \cos(e^x)$ (l) $A' = 8\sin c + 8c\cos c$

(m) $x' = \frac{2t - \sin t}{t^2 + \cos t}$ (n) $p' = \frac{1+2n}{3\sqrt[3]{(n+n^2)^2}}$ (o) $y' = \frac{-4}{(4x+2)^2}$

(p) $f' = (x+1)e^x$ (q) $t' = 2s\cos s^2 + 2\sin s\cos s$ (r) $b' = \frac{2e^t(\sin t - \cos t)}{\cos^2 t}$

Q10 (a) $y' = -(3x^2 + e^x)\sin(x^3 + e^x)$ (b) $z' = 6\cos 6xc)$ $m' = 4 + 4\ln x$

(d) $y' = \frac{x\cos x - \sin x}{x^2}$ (e) $h' = -2e^x$ (f) $r' = 20\, e^{5s}$

(g) $y' = -4t^6\cos t - 24t^5 \sin t$ (h) $P' = \frac{4(3x^5 - x^2)x^3 - (15x^4 - 2x)x^4}{(3x^5 - x^2)^2}$

(i) $y' = \frac{1-n}{n(n-\ln n)^2}$

(j) $y' = \frac{14\cos x + 7x^2\sin x}{\cos^2 x}$ (k) $t' = 12e^{3x} + 6\sin 3x$ (l) $A' = \ln c\cos c + \frac{\sin c}{c}$

(m) $x' = (3 - \cos t)\, e^{3t - \sin t}$ (n) $p' = \frac{\cos r}{4\sqrt[4]{\sin^3 r}}$ (o) $y' = \frac{-5(6x+2)}{(3x^2+2x)^2}$

(p) $f' = (x^2 + 2x)e^x$ (q) $a' = -2x\sin x^2 - 6\cos x\sin x$ (r) $x' = \frac{2^t\ln 2\sin t - 2^t\cos t}{\sin^2 t}$

Q11 (a) $y' = 2x\cos 2x + \sin 2x$ (b) $y' = \frac{2\sin x + x\cos x}{x\sin x}$ (c) $s' = (36x^4 + 12x)e^{2x^3}$

(d) $y' = \frac{2 - 3\ln x^2}{x^4}$ (e) $h' = e^{1+x}(\cos 2x + 2\sin 2x)$ (f) $r' = \frac{3e^{3x}(\sin 3x - \cos 3x)}{\sin^2 3x}$

(g) $y' = e^t(3t^2 + 6t)\cos(3t^2 e^t)$ (h) $P' = 10(\sin 2x - 2x)^4(\cos 2x - 1)$

(i) $y' = \frac{e^{2t}(2t - 1 - 2\cos t - \sin t}{(t - \cos t)^2}$

(j) $y' = \frac{\sin x^2 - 2x^2\cos x^2}{\sin^2 x^2}$ (k) $t' = 4e^x(x\cos x + x\sin x + \sin x)$

(l) $A' = 3\sin(6b^2 + 2b) + 3b(12b + 2)\cos(6b^2 + 2b)$

(m) $x' = \frac{(2t^5\cos t^2 + 4t^3\sin t^2)}{t^4\sin t^2}$ (n) $p' = \frac{1 + 2n\cos n^2}{2\sqrt{n + \sin n^2}}$ (o) $y' = \frac{\cos x + \sin x}{2\cos^2 x}\sqrt{e^x\cos x}$

(p) $f' = 2 - x^3 e^{6x}(4 - 6x)$ (q) $t' = \frac{6[\ln(\sin 2s)]^2}{\tan 2s}$ (r) $b' = \frac{-e^{5t}(10\cos t^2 - 4t\sin t^2}{\cos^2 t^2}$

Q12 (a) $p' = 3x^2 \sin 5x + 5x^3 \cos 5x$ (b) $b' = \frac{-\sin x - \ln x \cos x}{2^x}$

(c) $y' = 3(\cos t + 3t)^2 (3 - \sin t)$ (d) $y' = \ln 3x + 1$

(e) $h' = 4x \cos(x^2 + 3) + 4x^3 \sin(x^2 + 3)$ (f) $r' = \frac{1+2a\cos a^2}{a+\sin a^2}$

(g) $y' = -8t \sin(3t^2) - 24t^3 \cos(3t^2)$ (h) $P' = (4 - 28x)\cos(7x^2 - 2x)\sin(7x^2 - 2x)$

(i) $s' = \frac{\cos x - x\sin x}{x\cos x}$ (j) $y' = \frac{2x\cos x - x^2 \sin x}{\cos^2 x}$

(k) $h' = \cos x + \frac{1}{\cos^2 x}$ (l) $A' = 2^{-x^3}(3 - 9x^3 \ln x)$

(m) $z' = \frac{-4t^4 - 2t^2}{(4t^3 - 2t)^2}$ (n) $g' = \frac{\cos x - x\sin x}{2\sqrt{x\cos x}}$ (o) $x' = \frac{8\cos t - 4t\sin t}{t\cos t}$

(p) $p' = \frac{4n^3 + 5n^4}{3\sqrt[3]{(n^4 + n^5)^2}}$ (q) $y' = \frac{(8x+4)\cos 2x - 4\sin 2x}{(4x+2)^2}$

(r) $t' = 2\sin s\cos s\cos^2 s - 2s\sin s^2 \sin^2 s$ (s) $c' = \frac{10\sin b + 2b\cos b}{(\cos b)^6}$

(t) $f' = -\sin x\sin(4x - x^2) + (4 - 2x)\cos x\cos(4x - x^2)$ (u) $y' = \sin\theta$

Q13 Rising at 0.316 m/h

Q14 8.5 people per year

Q15 (a) –0.165 mg/L/h (b) $t = 1$ (c) 1.104 mg/L

Q16 175 kg/year (Julie was a whale.)

Q17 Falling $8200/year

Q51 $y' = \frac{1}{x}\sin(x^2 e{-}\cos 3x) + \ln x(\cos(x^2 e^{-\cos 3x}))(2x e^{-\cos 3x} - 3x^2 \sin 3x\, e^{-\cos 3x})$

Q52 $y' = x^x(1 + \ln x)$

Q53 $x' = -8\,e^{-0.04t}(0.04\cos 2t + 2\sin 2t)$; 12.84 m/s

Q61 432 per day

Q62 (a) $8(x^2 + 7x)^7(2x + 7)$ (b) $(2x + x^2)e^x$ (c) $y' = \frac{2x^2 - 4x + 5}{(x^2+5)^2}$

(d) $2r\ln r + r$ (e) $m' = \frac{1}{\cos^2 x}$ (f) $2(x + e^x)\cos(x^2 + 2e^x)$

(g) $-(x^3 + 3x^2)e^x \sin(x^3 e^x) + 1$ (h) $(3x^2 + 7)\sin^2 3x + (6x^3 + 42x)\sin 3x\cos 3x$

Q63 0.919 km/h

Q71 6.03 m/s upwards

Q72 (a) $y' = 2x\cos x + x^2 \sin x$ (b) $y' = \frac{1}{\cos^2 x}$ (c) $h' = -0.4e^{1-0.1t} + 3$

(d) $s' = \frac{\sin x - \sqrt{x}\cos x}{2\sqrt{x}\sin^2 x}$ (e) $y' = \frac{3\cos(3\ln x)}{x}$ (f) $y' = \frac{2\cos x}{x} - 2\sin x\ln x$

(g) $y' = \frac{2\sin 5x + 5x\cos 5x}{x\sin 5x}$ (h) $\frac{(2x-3)\cos(x^2-3x) - 4\ln 2 \times \sin(x^2-3x)}{2^{4x}}$

Q73 1.418 m/s in the positive direction

Q81 $x = 1$

Q82 (a) $y = 7 \times 10^x (\ln 10 \times \cos x - \sin x)$ (b) $A' = \frac{(x-2)\, e^x}{x^3} + \cos x$ (c) $y' = \frac{2\cos x}{3\sqrt[3]{sin^2 x}}$

(d) $y' = \frac{4x^6(24x+14)}{(14x+2)^2}$ (e) $h' = -4 \tan t$ (f) $k' = e^x (\cos x - \sin x)$

(g) $y' = 3 \cos x \cos (3x + 4) \times [\cos (3x + 4) + 6 \sin (3x + 4)]$ (h) $b' = \frac{\cos t + t \sin t}{t \cos t}$

Q83 0.0123/day