M1 Maths

C6-7 Other Derivatives

- the derivatives of sin x, cos x, a^x, e^x and ln x

Summary Lead-In Learn Solve Revise Answers

Summary

The derivative of sin x is cos x; the derivative of cos x is −sin x.

The derivative of a^x is $a^x \ln a$; the derivative of e^x is e^x; the derivative of ln x is $\frac{1}{x}$.

As with expressions of the form ax^n, if we need to differentiate sums of terms, we differentiate each term and add them. If an expression is multiplied by a constant, its derivative is multiplied by the same constant. Also we can use the chain, product and quotient rules in the same way as for polynomial functions etc.

Lead-In

This is an activity you can do to help you see where the results that are presented in this module come from.

1. Draw up a set of axes with the same scales on the x- and y-axes. Sketch the graph of y = sin x where x is in radians. Then, on the same axes, sketch a graph of the gradient of the original graph. This of course is the derived function. Then look at the derived function and try to guess its formula.
2. Do the same for y = cos x.
3. Do the same for $y = 2^x$.

Learn

Introduction

So far we have seen how to differentiate expressions of the form ax^n and compound functions involving such terms. In this module, we are going to extend the range of expressions we can differentiate.

Derivatives of sin *x* and cos *x*

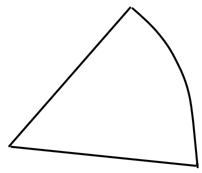
You should have discovered from the Lead-In that

$$\frac{d}{dx}\sin x = \cos x \qquad \text{and that } \frac{d}{dx}\cos x = -\sin x$$

You probably won't need to be able to prove these results. Just remember them.

This is provided that x is in radians. If x is in degrees, then the derivative of sin x is $\frac{2\pi}{360}\cos x$ and the derivative of cos x is $-\frac{2\pi}{360}\sin x$. But you will never use those.

[Remember that a radian is the angle for which the arc length is equal to the radius. A radian is about 57°. π radians = 180°.]



Practice

Q1 Differentiate the following.

(a) $p = \sin t$ (b) $p = -\sin t$ (e) $y = \cos x$ (f) $y = -\cos x$

Derivatives of a^x and e^x

You should have noticed from the Lead-In that the derivative of $y = 2^x$ looks like another exponential function very similar to $y = 2^x$. In fact, $\frac{d}{dx}2^x = 2^x \log_e 2$.

You met the number e in the module on exponential functions.

Other exponential functions have very similar derivatives. In general, $\frac{d}{dx}a^x = a^x \log_e a$.

You probably won't need to be able to prove this either; just remember it.

[As mentioned in the module on power and exponential functions, logs base e are often called natural logs and so $\log_e$ is usually written as ln (l for 'log', n for 'natural') and pronounced *lin*. The ln button is next to the log button on your calculator. Its inverse function is e^x as you would expect.]

A corollary of the fact that $\frac{d}{dx}a^x = a^x \log_e a$ is the fact that $\frac{d}{dx}e^x = e^x \log_e e$.

[A corollary is a second fact that follows on automatically from a result and is of comparable importance.]

Of course $\log_e e = 1$, so $\frac{d}{dx}e^x = e^x$. This is one of the reasons mathematicians use e as a base.

Practice

Q2 Differentiate the following.

(a) $h = 3^x$ (b) $P = 2^x$ (c) $s = 1.08^x$ (d) $t = e^x$

Derivative of ln x

If $y = \ln x$, $y' = \frac{1}{x}$. This is another case of the number e arising spontaneously.

Again, you probably won't need to prove this; just remember it.

Practice

Q3 Differentiate the following.

(a) $r = \ln t$ (b) $y = \log_e x$ (c) $y = \ln x$ (d) $m = \log_e k$

Summary

The new derivatives are listed in this table. Take the time to memorise them.

Primitive	**Derivative**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
a^x	$a^x \ln a$
e^x	e^x
$\ln x$	$^1/_x$

Sums and multiples

As with expressions of the form ax^n, if we need to differentiate a sum of terms, we just differentiate each term separately then combine them.

For example, the derivative of $\sin x - 4^x + 3x^4$ is $\cos x - 4^x \ln 4 + 12x^3$.

As $3\sin x = \sin x + \sin x + \sin x$, the derivative of $3\sin x$ is $3\cos x$. Likewise for other multiples of functions.

Practice

Q4 Differentiate the following.

(a) $s = \sin t + e^t$ (b) $s = \sin t + \cos t$ (c) $h = 2 \sin t$

(d) $y = \sin x - \cos x$ (e) $h = x - 3^x$ (f) $r = \ln t - t^2$

(g) $y = ½ \cos t$ (h) $P = 2^x \times 5$ (i) $s = 500 \times 1.08^x$

(j) $y = 3 \cos x - 2 \sin x$ (k) $t = 5e^x + \ln x$ (l) $A = 7x^2 + 5^x$

(m) $x = \frac{3 \ln t}{2}$ (n) $p = -2 \ln t - 4 \cos t$ (o) $y = 4 + 2 \ln x$

(p) $y = x - \cos x + 4e^x$ (q) $v = \frac{2\log_e t}{5}$ (r) $x = 4^t + 3 \sin t + \frac{1}{t}$

(s) $h = \sqrt{x} + \frac{e^x}{4}$ (t) $y = 5\frac{\sqrt{x}}{x^2} - 2 \cos x + e^x$ (u) $y = \cos x - x^4$

The Chain, Product and Quotient Rules can be used with exponential, logarithmic and trigonometric functions in just the same way that they are used with polynomial functions etc. Here are a few examples. You will get plenty more practice with this in the next module, C6-8.

Practice

Q5 Differentiate the following.

(a) $a = (\cos t)^6$ (b) $y = e^x \sin x$ (c) $h = \frac{\ln x}{\sin x}$

(d) $y = 5\sin x \cos x$ (e) $h = e^{\sin t}$ (f) $r = \frac{x}{\ln x}$

And finally, a few word problems using what you have learnt in this module.

Practice

Q6 The number of bacteria in a colony t hours after establishing itself is given by $n = 50 \times 2.3^t$.

(a) Find the rate at which it is growing when $t = 4$

(b) Use your answer to (a) to determine by roughly how many bacteria the colony will increase between $t = 4$ and $t = 4.1$

(c) Explain why the answer you get to (b) is only rough.

Q7 Find the equation of the tangent to the curve $y = 5 \ln x$ at $x = 8$.

Q8 The water level, h, in metres, in a reservoir is given by the formula

$h = 4 \sin t + \cos t$, where t is the time in days. How fast is the water rising or falling when $t = 14$?

Solve

Q51 You don't need to know the proof that $\frac{d}{dx} \ln x = \frac{1}{x}$, but it is not all that difficult. You might like to try if for a challenge. Basically, you start by letting $y = \ln x$, then rearrange to make x the subject, then find $\frac{dx}{dy}$ and finally take the reciprocal.

Q52 Use the index laws and trig identities to rearrange and differentiate the following.

(a) $y = (2e)^x$ (b) $y = e^{5x}$ (c) $s = \sin^2 t + \cos^2 t$ (d) $y = \sqrt{1 - cos^2 x}$

Revise

Revision Set 1

Q61 Differentiate

(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = 2^x$

(d) $h = e^t$ (e) $y = \ln x$ (f) $y = 2 \ln x - \cos x$

(g) $p = 5 \sin r + r^2$ (h) $y = \frac{1}{x^2} + 2 \cos x$ (i) $h = \sqrt{x} + \frac{e^x}{2}$

Revision Set 2

Q71 Differentiate

(a) $y = \cos x$ (b) $y = -\sin x$ (c) $y = 6^x$

(d) $r = e^x$ (e) $y = \ln t$ (f) $y = ½ \ln x - 4e^x$

(g) $p = 11 \cos r - 2r^5$ (h) $y = \frac{1}{\sqrt{x}} + \cos x$ (i) $h = 3\sqrt{x} + 3x^2 - 7^x$

Revision Set 3

Q81 Differentiate

(a) $y = e^x$ (b) $y = -\cos x$ (c) $y = 5^x$

(d) $h = \sin t$ (e) $y = 4 \ln x$ (f) $y = 2e^x - \sin x$

(g) $p = \cos r + 5(r^2 + 1)$ (h) $y = \frac{1}{x} + 2^x - \sin x$ (i) $h = 2\sqrt{x} + 3 \ln x$

Answers

Q1 (a) $p' = \cos t$ (b) $p' = -\cos t$ (e) $y' = -\sin x$ (f) $y' = \sin x$

Q2 (a) $h' = 3^x \ln 3$ (b) $P' = 2^x \ln 2$ (c) $s' = 1.08^x \ln 1.08$ (d) $t' = e^x$

Q3 (a) $r' = \frac{1}{t}$ (b) $y' = \frac{1}{x}$ (c) $y' = \frac{1}{x}$ (d) $m' = \frac{1}{k}$

Q4 (a) $s' = \cos t + e^t$ (b) $s' = \cos t - \sin t$ (c) $h' = 2 \cos t$

(d) $y' = \cos x + \sin x$ (e) $h' = 1 - 3^x \ln 3$ (f) $r' = \frac{1}{t} - 2t$

(g) $y' = -½ \sin t$ (h) $P' = 2^x \times 5 \ln 2$ (i) $s' = 500 \times 1.08^x \ln 1.08$

(j) $y' = -3 \cos x - 2 \cos x$ (k) $t' = 5e^x + \frac{1}{x}$ (l) $A' = 14x + 5^x \ln 5$

(m) $x' = \frac{3}{2t}$ (n) $p' = -\frac{2}{t} + 4 \sin t$ (o) $y' = \frac{2}{x}$

(p) $y' = 1 + \sin x + 4e^x$ (q) $v' = \frac{2}{5t}$ (r) $x' = 4^t \ln 4 + 3 \cos t - \frac{1}{t^2}$

(s) $h' = \frac{1}{2\sqrt{x}} + \frac{e^x}{4}$ (t) $y' = \frac{-15}{2\sqrt{x^5}} + 2 \sin x + e^x$ (u) $y' = -\sin x - 4x^3$

Q5 (a) $a' = -6(\cos t)^5 \sin t$ (b) $y' = e^x \sin x + e^x \cos x$ (c) $h' = \frac{{}^1/_x \sin x - \ln x \cos x}{(\sin x)^2}$

(d) $y' = 5(\cos^2 x - \sin^2 x)$ (e) $h' = e^{\sin t} \cos t$ (f) $r' = \frac{\ln x - 1}{(\ln x)^2}$

Q6 (a) 1165 per hour (b) 117

(c) because the population will be growing faster at t = 4.1 that the rate calculated

Q7 $y = 0.625x + 5.397$

Q8 falling by 0.444 m/day

Q52 (a) $y' = (2e)^x(1 + \ln 2)$ (b) $y' = 5e^{5x}$ (c) $s' = 0$ (d) $\cos x$

Q61 (a) $y' = \cos x$ (b) $y' = -\sin x$ (c) $y' = 2^x \ln x$

(d) $h' = e^t$ (e) $y' = \frac{1}{x}$ (f) $y' = \frac{2}{x} + \sin x$

(g) $p' = 5 \cos r + 2r$ (h) $y' = \frac{-2}{x^3} - 2 \sin x$ (i) $h' = \frac{1}{2\sqrt{x}} + \frac{e^x}{2}$

Q71 (a) $y' = -\sin x$ (b) $y' = -\cos x$ (c) $y' = 6^x \ln 6$

(d) $r' = e^x$ (e) $y' = \frac{1}{t}$ (f) $y' = \frac{1}{2x} - 4e^x$

(g) $p' = -11 \sin r - 10r^4$ (h) $y' = \frac{-1}{2\sqrt{x^3}} - \sin x$ (i) $h' = \frac{3}{2\sqrt{x}} + 6x - 7^x \ln 7$

Q81 (a) $y' = e^x$ (b) $y' = \sin x$ (c) $y' = 5^x \ln 5$

(d) $h' = \cos t$ (e) $y' = \frac{4}{x}$ (f) $y' = 2e^x - \cos x$

(g) $p' = -\sin r + 10r$ (h) $y' = \frac{-1}{x^2} + 2^x \ln 2 - \cos x$ (i) $h' = \frac{1}{\sqrt{x}} + \frac{3}{x}$