M1 Maths

C6-4 Other Relations

- calculus with relations other than between time, displacement and velocity
- generic *x*-*y* relations

Summary Learn Solve Revise Answers

Summary

The calculus ideas developed in previous modules in the context of time, displacement and velocity can be used with relations between other variables.

They can also be used with generic relations between unspecified variables (usually denoted x and y).

Learn

Calculus with Other Relations

So far, we have done calculus on the relationship between displacement, velocity and time. In the relation between displacement and time, time is generally the independent variable and displacement is the dependent variable.

Velocity is the rate at which displacement is changing as time changes. We say that it is the **rate of change** of displacement with respect to time. It is the ratio of the change in displacement to the change in time for an infinitely small change in time.

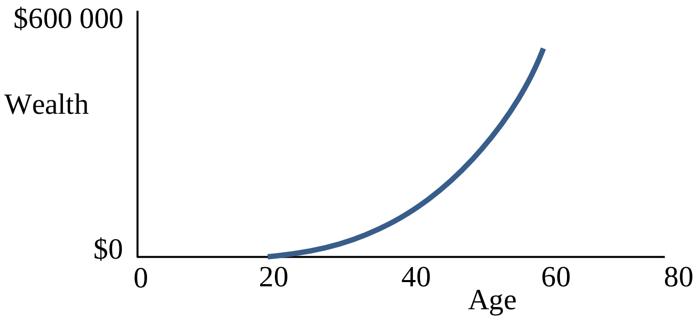
But the same ideas can be applied to any continuous relation. In any relation, there is an independent variable and a dependent variable, and, as the independent variable changes, the dependent variable will change too and we can calculate the rate of change of the dependent variable with respect to the independent variable. Again it will be the change in the dependent variable divided by the change in the independent variable for an infinitely small change in the independent variable.

Here is an example where the independent variable is time (someone's age) and the dependent variable is money (someone's wealth). We are looking at the rate of change of wealth with respect to age.

The relation between Edith Fanshaw's wealth and her age

Edith worked from age 20 to age 60 as a journalist. She didn't earn much at first, but managed to save a bit and her wealth started to grow. As time went on, her pay increased and so she managed to save more, so her wealth grew faster and so on until, at age 60, she was abducted by aliens and never heard of again.

A graph of the relation between her wealth and her age looks like this.



If w is her wealth in dollars and a is her age in years, then $w = 200a^2 - 3000a - 20000$ for $20 \leq a \leq 60$.

$\frac{dw}{da}$ is the rate of change of w with respect to age. It is the rate at which w is increasing. It would be measured in dollars/year.

As $w = 200a^2 - 3000a - 20000$

$\frac{dw}{da} = 400a - 3000$ (we just differentiate w with respect to a)

If we want to know the rate at which her wealth is growing when she turns 40, we just substitute 40 into the formula for $\frac{dw}{da}$.

$\frac{dw}{da} = 400 \times 40 - 3000 = 13\,000$

So, when she turns 40 her wealth is growing at $13 000 per year.

Note, however, that this doesn't mean that it will have grown by $13 000 by the time she turns 41. $13 000 per year is the rate of change of w when she turns 40. It is the gradient of the graph of w vs a at $a = 40$.

The rate of growth of w will increase through the year. By the time Edith gets to her 41st birthday, the rate of change will be $13 400 per year. So the increase in w over the year will be more than $13 000. $13 000 per year is the rate at which w is growing on her 40th birthday. It is an instantaneous rate of change. It is the gradient of the tangent to the graph at $a = 40$. The rate of change is greater when she's 41.

If we want to know how much her wealth increases between her 40th and 41st birthdays, we don't use calculus; instead we use the formula for w to calculate her wealth at 40 and her wealth at 41 and subtract. It comes to $13 200.

Practice

Q1 Mendip's wealth at age a is given by the formula $w = 120a^2 - 2000a - 5000$ for $20 \le a \le 60$. Find

(a) the formula for the rate of change of his wealth at any age, a

(b) the rate at which it is increasing when he is 30

(b) the increase between the age of 30 and the age of 35

Q2 Sally is retired and spending her money. Between the ages of 70 and 80, her wealth is given by the formula $w = -90a^2 + 1500a + 500\,000$. Find

(a) the formula for how fast her wealth is increasing at any age, a (it will be negative)

(b) the rate at which her wealth is decreasing when she is 76

(c) how much less money she has at 80 than at 70.

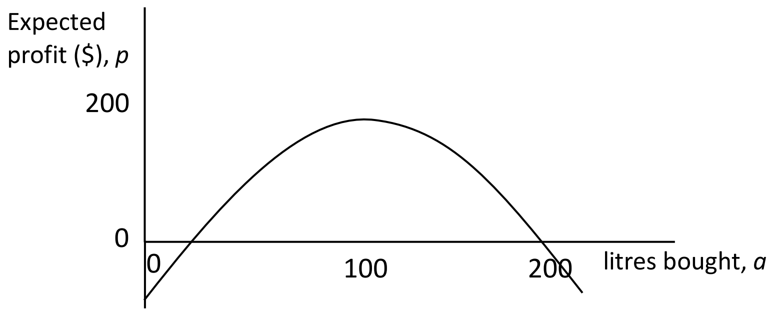
Here is another relation. In this one, neither variable is time. The independent variable is the amount of lemonade bought and the dependent variable is the profit made. We are looking at the rate of change of profit with respect to the amount of lemonade bought, i.e. the rate at which profit is increasing as the amount of lemonade bought increases or the increase in profit divided by the increase in amount bought for infinitely small increases in the amount bought.

Profit from selling lemonade

The skating club is running a lemonade stall at the local fete. They have to decide beforehand how much lemonade to buy, though, of course, they don't know exactly how much they will sell.

Past experience tells them that, if they buy a litres, the likely profit, p, will be given by $p = -50 + 5a - 0.025a^2$.

The relation between p and a looks like this:



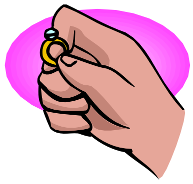
[It starts at –50 because they have to pay $50 to run the stall. Then it rises because the more lemonade they sell, the more profit they will make. But then it falls again because, if they buy too much, they will waste a lot.]

Roger suggests they buy 60 L. Rachel says they should buy more because the relation between p and a shows that p is increasing when $a = 60$.

She points out that, if $p = -50 + 5a - 0.025a^2$, then $\frac{dp}{da} = 5 - 0.05a$.
When $a = 60$, $\frac{dp}{da} = 5 - 0.05 \times 60 = 2$ dollars per litre. The gradient of the graph at $a = 60$ is 2. So they can expect to make \$2 more per extra litre bought.

So buying more that 60 would be a good idea. Of course, if they buy 150 litres, then the rate of increase of profit with respect to amount or lemonade bought, $\frac{dp}{da} = 5 - 0.05 \times 150 = -\$2.50$ per litre, so they would want to buy less than 150 L.

Practice



Q3 Large diamonds are worth more per carat than small ones. [1 carat is 0.2 g]. If the mass of a diamond in carats is m and the price in dollars is p, then $p = 2000\,m\sqrt{m}$.

(a) Find the formula for the rate of change of price with respect to mass

(b) At what rate is the price increasing (in dollars per carat) as the mass passes the 0.5 carat mark?

(c) How much more is a 0.6 carat diamond worth than a 0.5 carat diamond?

Q4 The area, A cm^2, of a circle is related to the radius, r cm, by $A = \pi r^2$. Find the rate of increase of A with respect to r

(a) Find the formula for the rate of change of A with respect to r

(b) Find the rate of change when $r = 6$

(c) Find the rate of increase when $r = 10$.

Q5 The Medieval Club is running a sausage sizzle. The profit, p, expected if s sausages are bought is given by $p = s - 0.005s^2$, where p is in dollars.

(a) Find the formula for the rate of change of expected profit with respect to the number of sausages bought.

(b) What would be the rate of change when $s = 0$?

(c) What would be the rate of change when $s = 150$?

Q6 Suppose the sausage formula in the previous question had been $p = 2s - 0.002s^2$.

(a) What would be the rate of change formula?

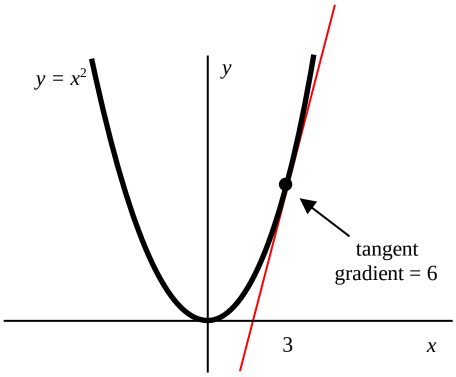
(b) What would be the rate of change when $s = 10$?

(c) What would be the rate of change when $s = 100$?

Generic Relations

So far, we have dealt with specific contexts. It is possible to use calculus to draw conclusions about generic function where the meaning of the variables isn't specified, like $y = 4x^2$. The conclusions drawn can then be applied to any situation that can be modelled by that formula.

For instance, if we wanted to know the rate of change (or gradient) of $y = x^2$, we just differentiate the function to get $\frac{dy}{dx} = 2x$. This gives us the gradient of the graph at any value of x. For instance, at $x = 3$, the gradient is $2 \times 3 = 6$. And the gradient of the tangent at $x = 3$ is 6.



Practice

Q7 (a) Find the gradient of $y = x^2$ at (i) $x = 5$ and (ii) $x = -1.5$

(b) Find $\frac{dy}{dx}$ if $y = 7x - x^2$

(c) Find the formula for the rate of change of y with respect to x if $y = 3x^4$

(d) Find gradient of the tangent to $y = x^3 - 2x$ at (i) $x = 4$ and (ii) $x = 0$

(e) Say whether the gradient of $y = x^3 - 4x$ is positive, negative or zero at (i) $x = \frac{1}{2}$ (ii) $x = {}^{2}/_{\sqrt{3}}$ (iii) $x = 2$

Solve

Q51 If $y = x^2 - 8x$, at what x-value is the graph horizontal (i.e. the gradient zero).

Q52 If $y = x^2 - 3x$, at what x-value is the gradient equal to –5).

Q53 Use your calculator to graph the function $y = 0.1x^3 - 0.2x^2 - 2x + 1$. Set the view window to $-8 \le x \le 8$ and $-5 \le y \le 5$. This should make the scales on the x and y axes about the same so the gradients look right. Copy the graph onto paper. Then, on the same axes sketch the gradient function. Do this by estimating by eye the gradient at several points along the graph, plotting the values of the gradient on the same axes and then joining them up to make a smooth curve. Then find the derivative of the original function and graph it on your calculator along with the original function. Use this to see how accurate your sketch of the gradient function is.

Q54 Do the same as in the last question, but with the function $y = 2.7^x$. What do you notice?

Q55 Set your calculator to radians and do the same again, but with the function $y = \sin x$. What do you notice?

Revise

Revision Set 1

Q61 Jojo's wealth at age a is given by the formula $w = 50a^2 - 2400a + 30\,000$ for $20 \le a \le 60$. Find

(a) the formula for the rate of change of her wealth at any age, a

(b) the rate at which her wealth is increasing when she is 32

(c) the increase between the age of 32 and the age of 35.

Q62 The volume of a sphere with radius r is given by $V = {}^{4\pi}/_3\, r^3$.

(a) Find a formula for the rate of change of volume with respect to radius.

(b) Find the rate of change of volume with respect to diameter when $r = 10$ cm.

(c) Find the increase in volume as the radius changes from 10 cm to 12 cm.

Q63 Find the gradient of the tangent to the curve $y = x^2 - 6x + 2$ at $x = 2$.

Answers

Q1	(a) $\frac{dw}{da} = 240a - 2000$	(b) \$5200 per year	(c) \$29 000
Q2	(a) $\frac{dw}{da} = -180a + 1500$	(b) \$12 180 per year	(c) \$120 000
Q3	(a) $\frac{dp}{dm} = 3000\sqrt{m}$	(b) \$2121.32/carat	(c) \$222.41
Q4	(a) $\frac{dA}{dr} = 2\pi r$	(b) 37.7 cm^2/cm	(b) 62.8 cm^2/cm
Q5	(a) $p = 1 - 0.01s$	(b) \$1 per sausage	(c) –50c per sausage
Q6	(a) $p = 2 - 0.004s$	(b) \$1.96 per sausage	(c) \$1.60 per sausage
Q7	(a) (i) 10 (ii) –3	(b) $\frac{dy}{dx} = 7 - 2x$	(c) $\frac{dy}{dx} = 12x^3$
	(d) (i) 10 (ii) –2	(e) (i) negative (ii) zero (iii) positive	

Q51 $x = 4$
Q52 $x = -1$
Q53 $y = 0.6x^2 - 2x - 3$
Q54 The gradient function is almost identical to the primitive function
Q55 The gradient function is identical to the primitive, except shifted ${}^{\pi}/_2$ to the left.
It is the function $y = \cos x$.

Q61 (a) $\frac{dw}{da} = 100a - 2400$ (b) \$800 / year (c) \$2850

Q62 (a) $\frac{dV}{dr} = 4\pi r^2$ (b) 1256 cm^3 / cm (c) 3049 cm^3

Q63 -2