M1 Maths

C6-3 Velocity by Rule

- differentiating at^n and sums of such terms by rule

Summary Lead-In Learn Solve Revise Answers

Summary

It is possible to differentiate functions of the form $s = at^n$ by a simple rule, thus side-stepping the techniques learnt in Module C6-2. You simply multiply by the index, n, then subtract 1 from the index. To differentiate an expression with more than one term, simply differentiate each term in turn.

Learn

Differentiation by Rule

There is actually a very quick and easy way to find the derivatives of functions of the form at^n, where a and n are real numbers without working them out algebraically.

All you do is multiply by the index to get nat^n,

then subtract 1 from the index to get nat^{n-1}.

So for example, to differentiate $s = 3t^5$,

we multiply by 5 to get $s = 15t^5$,

then we subtract 1 from the index to get $s = 15t^4$.

This is called differentiating by rule. Practise on the following questions. Although they are easy, you need to do quite a few because this needs to be something you can do almost without thinking.

Practice

Q1 Differentiate the following:

(a) $s = 4t^6$	(b) $s = t^3$	(c) $s = 2t^4$	(d) $s = t^2$
(e) $s = 2.5t^4$	(f) $s = t^{13}$	(g) $s = \frac{1}{4}t^8$	(h) $s = 7.5t^2$
(i) $s = 5t^6$	(j) $s = 0.1t^5$	(k) $s = \frac{1}{2}t^6$	(l) $s = 3t^3$

Why Does This Work

Let's differentiate $s = at^n$ algebraically as we did in Module C6-2.

$$\frac{\Delta s}{dt} = \frac{a(t+dt)^n - at^n}{dt}$$

$$= \frac{a(t^n+nt^{n-1}dt+\frac{1}{2}n(n-1)t^2dt^2+\cdots) - at^n}{dt}$$ (using the binomial expansion)

$$= \frac{at^n+ant^{n-1}dt+\frac{1}{2}an(n-1)t^{n-2}dt^2 \ldots -at^4}{dt}$$

$$= \frac{ant^{n-1}dt+\frac{1}{2}an(n-1)t^{n-2}dt^2 \ldots}{dt}$$

$$= ant^{n-1} + \frac{1}{2}an(n-1)t^{n-2}dt \ldots$$

$$\frac{ds}{dt} = \lim_{dt \to 0} \frac{\Delta s}{dt} = ant^{n-1} = nat^{n-1}$$

Note that the binomial expansion works only when n is a whole number, but it can be shown that the differentiation rule works for negative and fractional values of n – any real number. It also works for any real number value for a.

Non-whole-number Indices

As mentioned, this rule works not only for whole number indices like t^2 or t^6, but for any real number indices like t^1, t^0, t^{-3}, $t^{½}$, $t^{-3.71}$ and so on.

So, for example the derivative of t^{-2} is $-2t^{-3}$ and the derivative of t½ is $½t^{-½}$.

Practice

Q2 Find the derived function for each of the following:

(a) $s = t^{-6}$	(b) $s = t^{-1}$	(c) $s = 2t^{-4}$	(d) $s = t^1$
(e) $s = 2t^{½}$	(f) $s = t^{0.2}$	(g) $s = 3t^{-0.5}$	(h) $s = 5t^0$
(i) $s = 0.4t^{0.5}$	(j) $s = 0.1t^{3½}$	(k) $s = ½t^{¼}$	(l) $s = 3t^{-2.5}$

Fractions and Roots

Now that you can differentiate functions with negative and fractional powers of the variable, you can differentiate functions like $\frac{4}{t^2}$ and $\sqrt{t^3}$. You just use the index rules to convert them to the form at^n, then differentiate.

So, to differentiate $s = \frac{4}{t^2}$, we convert it to $s = 4t^{-2}$, then we differentiate that to get $s = -8t^{-3}$, then we convert it back to the form it was given in, i.e. $\frac{-8}{t^3}$.

The answer wouldn't be wrong if we left it as $s = -8t^{-3}$, but there is a tradition that we always give the derivative in the same form as the original function. Not doing so can lead to 6 minutes of bad luck. Besides, you will probably be expected to do so on tests.

Note that to differentiate $s = 4$, we can re-write this as $s = 4t^0$, so the derivative is $0 \times 4t^0$, which is 0. To differentiate $s = 4t$, we write it as $s = 4t^1$, so the derivative is $1 \times 4t^0$, which is 4. The derivative of any constant is zero, the derivative of at where a is a constant is a.

Practice

Q3 For each of the following, find $\frac{ds}{dt}$.

(a) $s = \frac{1}{t^2}$ (b) $s = \frac{3}{t^2}$ (c) $s = \frac{-4}{t^5}$ (d) $s = t$

(e) $s = 5t$ (f) $s = 7$ (g) $s = 4$ (h) $s = 5t^0$

(i) $s = -3$ (j) $s = \sqrt{t}$ (k) $s = \sqrt[4]{t}$ (l) $s = \sqrt{t^3}$

(m) $s = 2\sqrt[4]{t^5}$ (n) $s = \frac{1}{\sqrt{t}}$ (o) $s = \frac{4}{\sqrt[4]{t}}$ (p) $s = 2(\sqrt{t})^5$

(q) $s = \frac{1}{5\sqrt{t}}$ (r) $s = \frac{10}{3(\sqrt[5]{t})}$

Alternative Notations

$\frac{d}{dt}$ *Notation*

If $s = t^2$, then the derivative of s can be written $\frac{ds}{dt}$, but, as $s = t^2$, it can also be written as $\frac{dt^2}{dt}$.

We don't often use this notation, but we do use a variation on it: $\frac{d}{dt}\,t^2$, pronounced *d by dt of t squared*.

So $\frac{d}{dt}\,t^2$ means the derivative of t^2, which will be $2t$.

It doesn't actually matter whether the original function was $s = t^2$, $p = t^2$ or $a = t^2$, the derivative will always be $2t$, so not having the s in the expression isn't generally a problem.

Dash Notation

If $s = t^2$, then another notation for the derivative of s is s', pronounced *s-dash*. So $s' = 2t$.

So, if $s = t^2$, we can write the derivative of s as $\frac{ds}{dt}$, or as $\frac{d}{dt}\,t^2$ or as s'.

Practice

Q4 Find:

(a) $\frac{d}{dt}\sqrt{t^3}$ (b) $\frac{d}{dt}\,2\sqrt[4]{t^5}$ (c) $\frac{d}{dt}\frac{1}{\sqrt{t}}$ (d) $\frac{d}{dt}\frac{4}{\sqrt[4]{t}}$

Q5 Find s' if

(a) $s = 3t$ (b) $s = \frac{1}{t^2}$ (b) $s = \sqrt{t^5}$ (c) $s = 6$

Sums and Multiples of Terms

If you need to differentiate a function which consists of a number of terms, i.e. a number of expressions added and subtracted like $4t^2 + 2t$ or $7t - 5$, you just differentiate each in turn.

So, if $s = 4t^2 + 2t$, then $\frac{ds}{dt} = 8t + 2$.

If $s = 7t - 5$, then $\frac{ds}{dt} = 7$.

If $s = 4t^2 - t^3$, then $\frac{ds}{dt} = 8t - 3t^2$.

If $s = 3t^3 - 5t^2 + 7t - 10$, then $\frac{ds}{dt} = 9t^2 - 10t + 7$.

It follows from this that if you need to differentiate a function multiplied by a constant, then the derivative is just multiplied by the same constant.

So, as $\frac{d}{dt}\,t^2 = 2t$, $\frac{d}{dt}\,3t^2 = 6t$ (because it equals : $\frac{d}{dt}\,(t^2 + t^2 + t^2)$)

as $\frac{d}{dt} 5t^4 = 20t^3$, $\frac{d}{dt} 7 \times 5t^4 = 7 \times 20t^3, = 140t^3$

Practice

Q6 Find the derivatives of the following:

(a) $s = t^2 + 4t$ (b) $s = 4t^3 - 2t^2$ (c) $s = 6t^3 + \frac{3}{t^2}$

(d) $s = t^5 - 15t^3 - t^2 + 3t - 11$ (e) $s = 5t - 4$ (f) $s = 7 - \sqrt{t}$

(g) $s = -t + 4t^2$ (h) $s = \sqrt{t} + \frac{1}{\sqrt{t}}$ (i) $s = \sqrt[4]{t} + \sqrt[3]{t}$

Simplifying First

If we have to differentiate something like $s = (5t^2 + 1)(t - 5)$, we just expand first to get $5t^3 - 25t^2 + t - 5$. Differentiating this then gives $s' = 15t^2 - 50t + 1$.

If we have to differentiate something like $s = \frac{2t^3 + 5t^2}{t}$, we cancel first to get $2t^2 + 5t$. Differentiating this then gives $s' = 4t + 5$.

In the same way, if we have to differentiate $s = t^2\sqrt{t}$, we use the index laws to simplify this to $t^{2½}$, then differentiate to get $2½t^{1½}$ which is $2½t\sqrt{t}$.

Practice

Q7 Find the derivatives of the following:

(a) $s = 3t(6 - t^2)$ (b) $s = \frac{2t^3 + 5t^2}{t}$ (c) $s = (3t^2 + 4)(t - 5)$

(d) $s = 4t(t + 2)(t - 5)$ (e) $s = (t + 6)^2$ (f) $s = t^2\sqrt{t}$

(g) $s = t\sqrt[3]{t}$ (h) $s = \sqrt{t^3}\sqrt{t}$ (i) $s = \frac{\sqrt{t}}{t}$

(j) $s = \frac{t^2}{\sqrt{t}}$ (k) $s = 4t + 3t\sqrt{t^3}$ (l) $s = -3 - 5t^2\sqrt{t}$

So the question is: If functions can be differentiated so simply, why did we learn the time-consuming methods in C6-1 to C6-2?

Two reasons. The first is that calculus is a vast area of knowledge which underlies much of higher mathematics and it is worth taking a few hours to understand the concepts it is based on and how it works, rather than just memorising some rules without knowing why they work.

The second is that, not all functions are of the form $s = at^n$. Though there are rules for other types of functions (and you will meet dome later), occasionally you might need to find a velocity (or other rate of change) for a function for which you don't know a rule.

You will probably need to show that you can use the longer methods in an exam, but, apart from that, from here on, you won't use them a lot.

Solve

Q51 If $\frac{ds}{dt} = 12t^2$, find the formula for s.

Q52 If t is measured in radians,

then $\sin t = \frac{t^1}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots$

and $\cos t = \frac{t^0}{0!} - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots$

Show that, if $s = \sin t$, then $\frac{ds}{dt} = \cos t$.

Q53 If $s = 2^t$, find the velocity when $t = 3$ to 3 decimal places. [You won't be able to use the rule on this one and will need to go back to one of the techniques from Module C6-2.]

Revise

Revision Set 1

Q61 Differentiate the following by rule.

(a) $s = 5t^2$ (b) $s = 7.5t^4$ (c) $s = 6t^{-2}$

(d) $s = 4t^{½}$ (e) $s = \frac{5}{t^4}$ (f) $s = 5\sqrt{t}$

(g) $s = 7t$ (h) $s = 11$ (i) $s = 0$

Q62 Find s' if:

(a) $s = \frac{12}{\sqrt[4]{t^3}}$ (b) $s = 3t^3 - 4t + 8$ (c) $s = t + \sqrt{t}$

Q63 Find:

(a) $\frac{d}{dt}(t^2 + t)(t - 5)$ (b) $\frac{d}{dt}\frac{t^4+3t^2}{t^2}$ (c) $\frac{d}{dt}\frac{t\sqrt{t}+5t^2}{\sqrt{t}}$

Revision Set 2

Q71 Differentiate the following by rule.

(a) $s = 7t^2$ (b) $s = 1.5t^6$ (c) $s = t^{-3}$

(d) $s = 6t^{½}$ (e) $s = \frac{2}{t^2}$ (f) $s = 5\sqrt[3]{t}$

(g) $s = t$ (h) $s = -2$ (i) $s = 0$

Q72 Find s' if:

(a) $s = \frac{5}{\sqrt[4]{t^7}}$ (b) $s = 6t^4 - 4t^2 + 1$ (c) $s = 2t - 5\sqrt{t}$

Q73 Find:

(a) $\frac{d}{dt}(t^3 + 4)(t - 1)$ (b) $\frac{d}{dt}\frac{5t^3+3t^2}{t}$ (c) $\frac{d}{dt}\frac{t^4\sqrt{t}+5t^2}{10t\sqrt{t}}$

Revision Set 3

Q81 Differentiate the following by rule.

(a) $s = 2t^6$ (b) $s = -1.5t^3$ (c) $s = 3t^{-4}$

(d) $s = 8t^{1.4}$ (e) $s = \frac{1}{t}$ (f) $s = \sqrt[4]{t}$

(g) $s = -3t$ (h) $s = 1$ (i) $s = -½$

Q82 Find s' if:

(a) $s = \frac{1}{\sqrt{t^3}}$ (b) $s = t^2 + 5t - 8$ (c) $s = \frac{1}{t} + \sqrt{t}$

Q83 Find:

(a) $\frac{d}{dt}t^2(t + 1)(t - 5)$ (b) $\frac{d}{dt}\frac{t+3t^2}{t^2}$ (c) $\frac{d}{dt}\frac{t\sqrt{t}+5t^2}{2t^2\sqrt{t}}$

Answers

Q1 (a) $\frac{ds}{dt} = 24t^5$ (b) $\frac{ds}{dt} = 3t^2$ (c) $\frac{ds}{dt} = 8t^3$ (d) $\frac{ds}{dt} = 2t$

(e) $\frac{ds}{dt} = 10t^3$ (f) $\frac{ds}{dt} = 13t^{12}$ (g) $\frac{ds}{dt} = 2t^7$ (h) $\frac{ds}{dt} = 15t$

(i) $\frac{ds}{dt} = 30t^5$ (j) $\frac{ds}{dt} = 0.5t^4$ (k) $\frac{ds}{dt} = 3t^5$ (l) $\frac{ds}{dt} = 9t^2$

Q2 (a) $\frac{ds}{dt} = -6t^{-7}$ (b) $\frac{ds}{dt} = -t^{-2}$ (c) $\frac{ds}{dt} = -8t^{-5}$ (d) $\frac{ds}{dt} = 1$

(e) $\frac{ds}{dt} = t^{-½}$ (f) $\frac{ds}{dt} = 0.2t^{-0.8}$ (g) $\frac{ds}{dt} = -1.5t^{-1.5}$ (h) $\frac{ds}{dt} = 0$

(i) $\frac{ds}{dt} = 0.2t^{-0.5}$ (j) $\frac{ds}{dt} = 0.35t^{2½}$ (k) $\frac{ds}{dt} = {}^1/_8 t^{-3/4}$ (l) $\frac{ds}{dt} = -7.5t^{-3.5}$

Q3 (a) $\frac{ds}{dt} = \frac{-2}{t^3}$ (b) $\frac{ds}{dt} = \frac{-6}{t^3}$ (c) $\frac{ds}{dt} = \frac{-20}{t^6}$ (d) $\frac{ds}{dt} = 1$

(e) $\frac{ds}{dt} = 5$ (f) $\frac{ds}{dt} = 0$ (g) $\frac{ds}{dt} = 0$ (h) $\frac{ds}{dt} = 0$

(i) $\frac{ds}{dt} = 0$ (j) $\frac{ds}{dt} = \frac{1}{2\sqrt{t}}$ (k) $\frac{ds}{dt} = \frac{1}{4\sqrt[4]{t^3}}$ (l) $1.5\sqrt{t}$

(m) $2.5\sqrt[4]{t}$ (n) $\frac{-1}{2\sqrt{t^3}}$ (o) $\frac{-1}{\sqrt[4]{t^5}}$ (p) $5t\sqrt{t}$

(q) $\frac{-1}{10\sqrt{t^3}}$ (r) $\frac{-2}{3\sqrt[5]{t^6}}$

Q4 (a) $\frac{3\sqrt{t}}{2}$ (b) $\frac{5\sqrt[4]{t}}{2}$ (c) $\frac{-1}{2\sqrt{t^3}}$ (d) $\frac{-1}{\sqrt[4]{t^5}}$

Q5 (a) $s' = 3$ (b) $s' = \frac{-2}{t^3}$ (b) $s' = \frac{5\sqrt{t^3}}{2}$ (c) $s' = 0$

Q6 (a) $\frac{ds}{dt} = 2t + 4$ (b) $\frac{ds}{dt} = 12t^2 - 4t$ (c) $\frac{ds}{dt} = 18t^2 - \frac{6}{t^3}$

(d) $\frac{ds}{dt} = 5t^4 - 45t^2 - 2t + 3$ (e) $\frac{ds}{dt} = 5$ (f) $\frac{ds}{dt} = -\frac{1}{2\sqrt{t}}$

(g) $\frac{ds}{dt} = -1 + 8t$ (h) $\frac{ds}{dt} = \frac{1}{2\sqrt{t}} - \frac{1}{2\sqrt{t^3}}$ (i) $\frac{ds}{dt} = \frac{1}{4\sqrt[4]{t^3}} + \frac{1}{3\sqrt[3]{t^2}}$

Q7 (a) $\frac{ds}{dt} = 18 - 9t^2$ (b) $\frac{ds}{dt} = 4t + 5$ (c) $\frac{ds}{dt} = 9t^2 - 30\text{t} + 4$

(d) $\frac{ds}{dt} = 12t^2 - 24t - 40$ (e) $\frac{ds}{dt} = 2t + 12$ (f) $\frac{ds}{dt} = \frac{5\sqrt{t^3}}{2}$

(g) $\frac{ds}{dt} = \frac{4\sqrt[3]{t}}{3}$ (h) $\frac{ds}{dt} = 2t$ (i) $-\frac{1}{2\sqrt{t^3}}$

(j) $\frac{ds}{dt} = \frac{3\sqrt{t}}{2}$ (k) $\frac{ds}{dt} = 4 + \frac{15\sqrt{t^3}}{2}$ (l) $-12.5\sqrt{t^3}$

Q51 $s = 4t^3$

Q53 5.545

Q61 (a) $\frac{ds}{dt} = 10t$ (b) $\frac{ds}{dt} = 30t^3$ (c) $\frac{ds}{dt} = -12t^{-3}$

(d) $\frac{ds}{dt} = 2t^{-½}$ (e) $\frac{ds}{dt} = -\frac{20}{t^5}$ (f) $\frac{ds}{dt} = \frac{2.5}{\sqrt{t}}$

(g) $\frac{ds}{dt} = 7$ (h) $\frac{ds}{dt} = 0$ (i) $\frac{ds}{dt} = 0$

Q62 (a) $s' = \frac{-9}{\sqrt[4]{t^7}}$ (b) $s' = 9t^2 - 4$ (c) $s' = 1 + \frac{1}{2\sqrt{t}}$

Q63 (a) $3t^2 - 8t - 5$ (b) $2t$ (c) $1 + \frac{15\sqrt{t}}{2}$

Q71 (a) $\frac{ds}{dt} = 14t$ (b) $\frac{ds}{dt} = 9t^5$ (c) $\frac{ds}{dt} = -3t^{-4}$

(d) $\frac{ds}{dt} = 3t^{-½}$ (e) $\frac{ds}{dt} = -\frac{4}{t^3}$ (f) $\frac{ds}{dt} = \frac{5}{3\sqrt[3]{t^2}}$

(g) $\frac{ds}{dt} = 1$ (h) $\frac{ds}{dt} = 0$ (i) $\frac{ds}{dt} = 0$

Q72 (a) $s' = \frac{-35}{4\sqrt[4]{t^{11}}}$ (b) $s' = 24t^3 - 8t$ (c) $s' = 2 - \frac{5}{2\sqrt{t}}$

Q73 (a) $4t^3 - 3t^2 + 4$ (b) $10t + 3$ (c) $\frac{3t^2}{10} - \frac{1}{4\sqrt{t}}$

Q81 (a) $\frac{ds}{dt} = 12t^5$ (b) $\frac{ds}{dt} = -4.5t^2$ (c) $\frac{ds}{dt} = -12t^{-5}$

(d) $\frac{ds}{dt} = 11.2t^{0.4}$ (e) $\frac{ds}{dt} = -\frac{1}{t^2}$ (f) $\frac{ds}{dt} = \frac{1}{4\sqrt[4]{t^3}}$

(g) $\frac{ds}{dt} = -3$ (h) $\frac{ds}{dt} = 0$ (i) $\frac{ds}{dt} = 0$

Q82 (a) $s' = \frac{-3}{2\sqrt{t^5}}$ (b) $s' = 2t + 5$ (c) $s' = -\frac{1}{t^2} + \frac{1}{2\sqrt{t}}$

Q83 (a) $4t^3 - 12t^2 - 10t$ (b) $-\frac{1}{t^2}$ (c) $-\frac{1}{2t^2} - \frac{5}{4\sqrt{t^3}}$