M1 Maths

C6-2 Velocity Algebraically

- velocity at a given time using reducing intervals
- velocity at given time using an unspecified interval, *dt*
- velocity at any time

Summary   **Learn**   **Solve**   **Revise**   **Answers**

Summary

If we have a formula for the displacement, s, it is possible to find the exact instantaneous velocity at a given time. We find the average velocity between that time and say 2 s after that time. This will be an approximation, but not exact. We can make the approximation closer by taking the average velocity over a shorter time interval, say 1 second. Then we can keep repeating the calculation with shorter and shorter time intervals. Eventually, we will see the value that our answers are approaching. That is the exact velocity.

We can short-cut a lot of the calculation by calculating the velocity between the time we want and a time dt later. Then we can find what this expression tends towards as dt approaches 0.

We can get a formula that will give us velocity at any time if we use the variable t in the calculation rather than using a specific value.

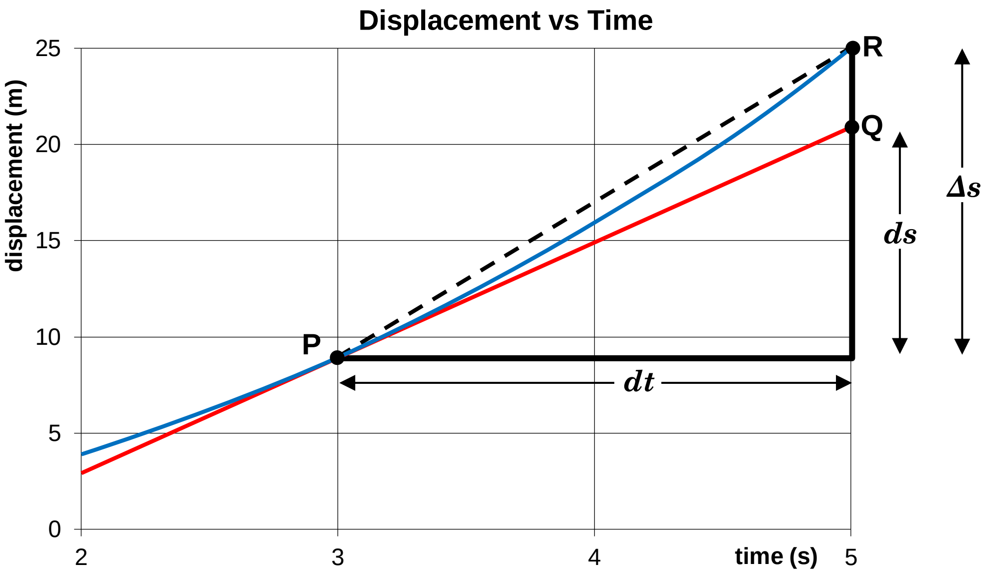
Learn

In Module C6-1 we were given the relations between distance and time as graphs. If we have them as formulae we can work out the instantaneous velocity at a given time more accurately than we can from a graph. In this module we will learn how to do that. We will also learn how to derive a formula for velocity in terms of time that can then be used to find the velocity at any time.

Step 1: Velocity at a given time using reducing intervals

Suppose we have the relation between displacement and time given by the formula $s = t^2$ where s is the displacement from a given point in metres and t is the time in seconds. [Admittedly, d would be a more logical abbreviation for *displacement*, but in calculus, we use d for *difference*, as you will see shortly. So we use s for *displacement*.]

The relation would look like the blue curve in the graph below.



Suppose we wanted to know the velocity at $t = 3$. We would draw a tangent to the curve at that point, the red line on the graph.

Now the velocity at $t = 3$ will be the gradient of the tangent PQ, so that is what we need to find. Let's call the difference in time between P and Q dt. And let's call the difference in displacement between P and Q ds, as shown on the graph.

The gradient we want is $\frac{ds}{dt}$.

Now dt is clearly 2. To find ds, however, we need to know the value of s at P and at Q. P is easy – it's on the curve $s = t^2$, so it's just 3^2.

But Q is not on the curve and we don't know the formula for the tangent. So the best we can do is find the coordinates of R and get the gradient of PR, the black dotted line, instead. Clearly the gradient of PR will be greater than that of PQ, but it will be a reasonable approximation.

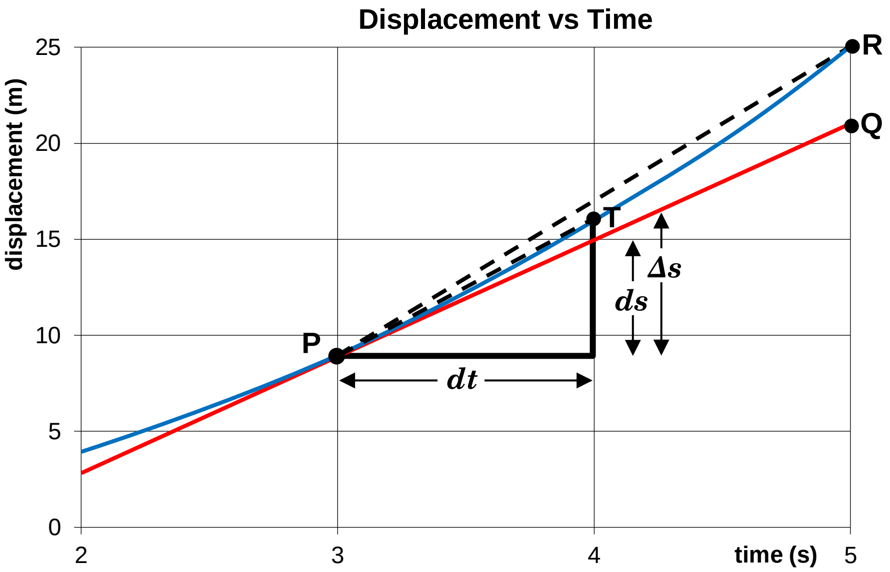
Let's call the difference in s between P and R Δs as shown on the graph.

$\frac{\Delta s}{dt}$ is our approximation for $\frac{ds}{dt}$.

So what is $\frac{\Delta s}{dt}$? $\Delta s = 5^2 - 3^2 = 16$. $dt = 2$. Therefore $\frac{\Delta s}{dt} = \frac{16}{2} = 8$

So $\frac{ds}{dt}$ is approximately 8, though we know it is a bit less than 8.

It is possible to make $\frac{\Delta s}{dt}$ a better approximation for $\frac{ds}{dt}$. We can do this by reducing dt. If we use PT instead of PR (see the next diagram), then $dt = 1$. As we can see, the gradient of PT (the shorter black dotted line) is less than the gradient of PR, though still more than the gradient of PQ. So it will give a better approximation to $\frac{ds}{dt}$.



Using PT, $\Delta s = 4^2 - 3^2 = 7$. $dt = 1$. Therefore $\frac{\Delta s}{dt} = \frac{7}{1} = 7$

The reason Δs is not equal to ds is that the graph curves away from the tangent. Between $t = 3$ and $t = 5$, it curves quite a bit and so gets a long way from the tangent. Between $t = 3$ and $t = 4$, it curves less and so the error is less. If we make dt smaller still, the graph will curve even less. Between $t = 3$ and $t = 3.1$, the graph looks almost straight and therefore will stay very close to the tangent. Let's do the calculation for $dt = 0.1$.

With $dt = 0.1$, $\Delta s = 3.1^2 - 3^2 = 9.61 - 9 = 0.61$. $dt = 0.1$. Therefore $\frac{\Delta s}{dt} = \frac{0.61}{0.1} = 6.1$

This is less than 7, but it will still be more than $\frac{ds}{dt}$. We are getting more accurate.

Of course, the smaller we make dt, the closer $\frac{\Delta s}{dt}$ will come to $\frac{ds}{dt}$, the gradient of the tangent.

Let's make $dt = 0.01$. Now $\frac{\Delta s}{dt} = \frac{3.01^2 - 3^2}{0.01}$ which is $\frac{9.0601 - 9}{0.01}$ which is 6.01.

Let's make $dt = 0.001$. Now $\frac{\Delta s}{dt} = \frac{3.001^2 - 3^2}{0.001}$ which is $\frac{9.006001 - 9}{0.001}$ which is 6.001.

Clearly, each time we stick another zero after the decimal point in dt, we get one more zero after the 6 in the answer. Theoretically we could keep going for ever and the answer will get closer and closer to 6.

But it will never go past 6. 6 is the **limit** to how far it can go. For any positive value for dt, the velocity will never actually reach exactly 6. If we could make dt equal to exactly 0, then the velocity might come to exactly 6. But then of course $\frac{\Delta s}{dt}$ would be $\frac{0}{0}$ which isn't defined.

In fact the velocity at $t = 3$ is exactly 6. But we can't show that by setting $dt = 0$. We just have to say that the limit as dt approaches zero of $\frac{\Delta s}{dt}$ is 6 and so $\frac{ds}{dt} = 6$. This is a subtle point, but an important point.

If we use function notation for velocity which is a function of time $v(t)$, we can write the statement in mathematical shorthand like this: $v(3) = \frac{ds}{dt} = \lim_{dt \to 0} \frac{\Delta s}{dt} = 6$.

Practice

Q1 Assume *velocity* = *time*^{2} ($v = t^2$). Use the technique above to find the velocity at $t = 2$. Do the calculation for $dt = 1$, 0.1, 0.01 and 0.001, then determine what the limit is.

Q2 Assume $v = t^2$. Use the technique above to find the velocity at $t = 4$. Do the calculation for $dt = 1$, 0.1, 0.01 and 0.001, then determine what the limit is.

Step 2: Velocity at a given time using an unspecified interval, *dt*

There is actually a much quicker way of doing this. Instead of picking a few different values for dt and doing the calculation for each, we can just leave it as the variable dt and deal with it algebraically. Let's go back to the original calculation – finding the velocity at $t = 3$ when $s = t^2$.

Instead of letting dt be 0.1, so $\frac{\Delta s}{dt} = \frac{3.1^2 - 3^2}{0.1}$,

let's leave dt as dt. Now $\frac{\Delta s}{dt} = \frac{(3+dt)^2 - 3^2}{dt}$.

Expanding the bracket we get $\frac{\Delta s}{dt} = \frac{9+6dt+dt^2 - 9}{dt}$.

$$= \frac{6dt+dt^2}{dt}$$

$$= 6 + dt$$

Now clearly, as dt approaches 0, $6 + dt$ approaches $6 + 0$, which is 6.

$$v(3) = \frac{ds}{dt} = \lim_{dt \to 0} \frac{\Delta s}{dt} = 6.$$

Easier, eh?

Practice

Q3 (a) Assume $s = t^2$. Use the dt method to find the velocity at $t = 4$.

(b) Assume $s = t^2$. Use the dt method to find the velocity at $t = 12$.

(c) Assume $s = t^2$. Use the dt method to find the velocity at $t = 1$.

Step 3: Velocity at any time

Up until now, we have found the velocity at particular times (for particular values for t). If we then wanted the velocity at a different time, we have to repeat the calculation.

But it is just as easy to find a formula for velocity in terms of time (the relation between velocity and time). So in general we do this, then substitute for the times we need to know about.

What we do is, rather than using a particular number for t before doing the algebra, we just leave it as the variable t and do the algebra with t rather than the number.

Suppose $s = t^2$. On the left below is the working to find the velocity at $t = 3$; on the right is the working to find a general formula for the velocity at any time, t.

$$\frac{\Delta s}{dt} = \frac{(3+dt)^2 - 3^2}{dt}.$$

$$= \frac{9+6dt+dt^2 - 9}{dt}.$$

$$= \frac{6dt+dt^2}{dt}$$

$$= 6 + dt$$

$$v(3) = \frac{ds}{dt} = \lim_{dt \to 0} \frac{\Delta s}{dt} = 6.$$

$$\frac{\Delta s}{dt} = \frac{(t+dt)^2 - t^2}{dt}$$

$$= \frac{t^2+2tdt+dt^2 - t^2}{dt}.$$

$$= \frac{2tdt+dt^2}{dt}$$

$$= 2t + dt$$

$$v(t) = \frac{ds}{dt} = \lim_{dt \to 0} \frac{\Delta s}{dt} = 2t$$

This is much more powerful. In one fell swoop we have found a formula that can give us the velocity at any time. If $t = 7.6$, then we just sub 7.6 into $v = 2t$ to get $v = 15.2$.

This process of starting with a formula for s and deriving a formula for v is called **differentiating**. The formula we started with, $s = t^2$, is called the **primitive function** and the formula we derived, $s = 2t$, is called the **derived function** or simply the **derivative**. The derived function can also be called the **gradient function** because it gives us the gradient of the primitive function at any value of the independent variable. You will need to be familiar with all these alternative words.

Using function notation, if $s(t) = t^2$, $\frac{\Delta s}{dt} = \frac{(t+dt)^2 - t^2}{dt}$

But the displacement won't always be equal to t^2: it could be t^3 or $4t^4$ or $2t^2 - 5t$, and so on.

For a general function $s(t)$, $\frac{\Delta s}{dt} = \frac{s(t+dt) - s(t)}{dt}$ and

$$v(t) = \frac{ds}{dt} = \lim_{dt \to 0} \frac{\Delta s}{dt} = \lim_{dt \to 0} \frac{s(t+dt) - s(t)}{dt}$$

$$v(t) = \frac{ds}{dt} = \lim_{dt \to 0} \frac{s(t+dt) - s(t)}{dt}$$

This is an important formula because it allows us to find the derivative of any function (as long as we can do the algebra). For instance, if $s(t) = 5t^2 + t$,

$$\frac{\Delta s}{dt} = \frac{s(t+dt) - s(t)}{dt}$$

$$= \frac{[5(t+dt)^2 + (t+dt)] - [5 \times t^2 + t]}{dt}.$$

$$\frac{\Delta s}{dt} = .\frac{[5(t^2 + 2dt + dt^2) + (t+dt)] - [5t^2 + t]}{dt}$$

$$= \frac{[5t^2 + 10tdt + 5dt^2) + (t+dt)] - [5t^2 + t]}{dt}$$

$$= \frac{[5t^2 + t + (10t+1)dt + 5dt^2] - [5t^2 + t]}{dt}$$

$$= \frac{(10t+1)dt + 5dt^2}{dt}$$

$$= 10t + 1 + 5dt$$

$$v(t) = \frac{ds}{dt} = \lim_{dt \to 0} \frac{\Delta s}{dt} = 10t + 1$$

Practice

Q4 If $s = t^2$, find $\frac{ds}{dt}$.

Q5 If $s = 2t^2$, find $\frac{ds}{dt}$.

Q6 Differentiate $s = 4t$

Q7 Find the derivative of $2t^2 + 3t$.

Q8 If $s = 5t + 8$, find the function for velocity.

Q9 Find the gradient function for $s = t^2 - t$.

Q10 If the primitive function is $s = t^2 - 6$, what is the derived function?

Q11 Differentiate $s = 6 - t^2$.

Q12 Find the derived function for $s = 4t^2 + t + 5$

Q13 If $s = t^2 - 3t - 1$, find $\frac{ds}{dt}$.

If the formula for s involves a higher power of t, say $s = 5t^4$, we can expand long-hand or we can use the binomial expansion (Module A6-1) to do the expanding.

Suppose $s = 5t^4$.

$$\frac{\Delta s}{dt} = \frac{5(t+dt)^4 - 5t^4}{dt}$$

$$= \frac{5(t^4+4t^3dt+6t^2dt^2+4tdt^3+dt^4) - 5t^4}{dt}$$

$$= \frac{5t^4+20t^3dt+30t^2dt^2+20tdt^3+5dt^4-5t^4}{dt}$$

$$= \frac{20t^3dt+30t^2dt^2+20tdt^3+5dt^4}{dt}$$

$$= 20t^3 + 30t^2dt^2 + 20tdt^3 + 5dt^4$$

$$v(t) = \frac{ds}{dt} = \lim_{dt\to 0}\frac{\Delta s}{dt} = 20t^3$$

Now we could have saved ourselves time there by only working out the first three terms of the expansion like this

$$= \frac{5(t^4+4t^3dt+62^2dt^2\ldots) - 5t^4}{dt}$$

because none of the subsequent terms will make any difference to the answer. This will always be the case when expanding powers.

Practice

Q14 Find $\frac{ds}{dt}$ if (a) $s = t^3$ (b) $s = t^5$ (c) $s = 4t^8$

Solve

Q51 You have found velocities for $s = t^2$, $s = t^3$ and $s = t^5$. It is possible to generalise this to any whole number power of t, $s = t^n$. Derive a formula for the velocity at $t = 1$ if $s = t^n$.

Q52 You also found the velocity for $s = 4t^8$. Derive a formula for the velocity at $t = 2$ if $s = at^n$, where a is a constant.

Q53 If the velocity of a rocket is given by $v = 3t^2$, find the acceleration, a, when $t = 5$.

Revise

Revision Set 1

Q61 Assume $v = 4t^2$. Using dt = 1, 0.1, 0.01 and 0.001, calculate the velocity at $t = 5$, then determine what the limit is.

Q62 Assume $s = t^2$. Use the dt method to find the velocity at $t = 4$.

Q63 If $s = 5t^2 - 3t$, find $\frac{ds}{dt}$

Q64 Differentiate $s = 2t^{10}$

Revision Set 2

Q71 Assume $v = 2t^2$ from $t = 0$ to $t = 10$. Using dt = 1, 0.1, 0.01 and 0.001, calculate the velocity at $t = 6$, then determine what the limit is.

Q72 Assume $s = t^2$. Use the dt method to find the velocity at $t = 10$.

Q73 If $s = 2t^2 + 6$, find $\frac{ds}{dt}$

Q74 Differentiate $s = 5t^7$

Revision Set 3

Q81 Assume $v = \frac{1}{2}t^2$ from $t = 0$ to $t = 6$. Using $dt = 1$, 0.1, 0.01 and 0.001, calculate the velocity at $t = 2$, then determine what the limit is.

Q82 Assume $s = t^2$. Use the short-cut dt method to find the velocity at $t = 5$.

Q83 If $s = 4t^2 + t$, find $\frac{ds}{dt}$

Q84 Differentiate $s = -t^8$

Answers

Q1	4	Q2	8	Q3	(a) 8	(b) 24	(c) 2		

Q4	$\frac{ds}{dt} = 2t$	Q5	$\frac{ds}{dt} = 4t$	Q6	$\frac{ds}{dt} = 4$
Q7	$\frac{ds}{dt} = 4t + 3$	Q8	$v = 5$	Q9	$\frac{ds}{dt} = 2t - 1$
Q10	$\frac{ds}{dt} = 2t$	Q11	$\frac{ds}{dt} = -2t$	Q12	$\frac{ds}{dt} = 8t + 1$
Q13	$\frac{ds}{dt} = 2t - 3$				

Q14 (a) $3t^2$ (b) $5t^4$ (c) $32t^7$

Q51	$v = n$	Q52	$v = 2^{n-1}na$	Q53	30		
Q61	40	Q62	8	Q63	$10t - 3$	Q64	$20t^9$
Q71	24	Q72	20	Q73	$4t$	Q74	$35t^6$
Q81	2	Q82	8	Q83	$8t + 1$	Q84	$-8t^7$