M1 Maths

C6-14 Optimisation

- finding the value of a control quantity which gives the optimum value of an objective quantity

Summary Learn Solve Revise Answers

Summary

Optimisation means finding the value of a control quantity which will give the optimum (highest or lowest) value for an objective quantity.

Optimisation can be performed by graphing the relation between control and objective quantities (the objective function) and finding the maximum or minimum. Or it can be done using calculus. This module is concerned with the calculus approach.

Using calculus, we find the objective function, then equate its derivative to zero and solve for the value of the control quantity.

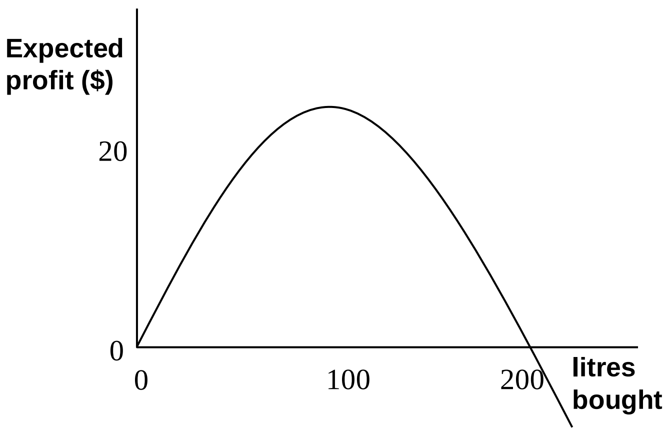
If there are two independent variables in the objective function, we eliminate one using the constraint provided in the question, then proceed as before.

Learn

Introduction

In Module C6-4 (Other Relations) we looked at the relation between expected profit and the number of litres of lemonade bought for a stall at the local fete.

Obviously, if no lemonade is bought, no profit will be made. As more is bought, more profit can be made. But there is a limit to how much is likely to be sold, so, if too much is bought, the profit will reduce and may even become negative. The relation might be something like $p = 5a - 0.025a^2$ where p is the expected profit in dollars and a is the number of litres bought. The graph would then look like this:



We would be interested in maximising the expected profit by choosing the value of a which gives the highest value for p. This would be an example of optimisation.

In this relation, a, the independent variable, is the quantity we can control. It is called the **control quantity**. p, the dependent variable, is the quantity we wish to optimise. It is called the **objective quantity**, because optimising it is the objective of the exercise. The formula for the objective quantity, $p = 5a - 0.025a^2$, is called the **objective function**.

Note that the optimum value may be a maximum value in the case of something like profit, or it could be the minimum value in the case of something like costs or time taken to do a job.

In an optimisation problem we need to use the objective function to find the value of the control quantity which gives the optimum value for the objective quantity.

There are two ways to do this.

The first is to graph the objective function and use GSolve to find the optimum value (which will be a maximum or a minimum). You already know how to do this.

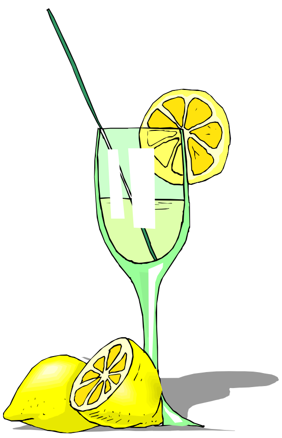
The second is to use calculus. Calculus was the only option before graphics calculators. Because it takes more effort than graphing, calculus, as a method, is, in a sense, obsolete. But it is still part of most maths courses and so probably should be learnt. Like other things which are obsolete, it is good practice at some of the component skills and reinforcement of some of the component concepts, so it isn't a complete waste of time.

In many optimisation problems, the objective function has to be found and this is a major part of the problem. This cannot be done on the calculator.

Optimisation Using Calculus

The optimum value will always be a maximum or a minimum of the objective function. In both cases the gradient will be zero. So, to find the optimum value, we differentiate the objective function, equate the derivative to zero and solve the equation.

In the case of the lemonade, the objective function is $p = 5a - 0.025a^2$, so the derivative is $\frac{dp}{da} = 5 - 0.05a$ and our equation is $0 = 5 - 0.05a$. Solving, we get $a = 100$. So buying 100 L of lemonade gives the best expected profit.



If we want to know the profit, we sub 100 for a in the objective function to get $p = 5 \times 100 - 0.025 \times 100^2 = 250$. So the expected profit if we buy 100 L is $250.

Practice

Answer the following questions using the calculus method described above.

Q1 The relation between expected profit, p, and number of litres of lemonade bought, a, is $p = 4a - 0.05a^2$. How much lemonade should be bought to get the maximum expected profit?

Q2 The relation between expected profit, p, and number of litres of lemonade bought, a, is $p = 10a - 0.005a^3$. How much lemonade should be bought to maximise the expected profit? What will the expected profit be if that much is bought?

Q3 The relation between expected profit and number of sausages bought for a sizzle is $p = s - 0.005s^2$, where p is the expected profit and s is the number of sausages bought. Find the maximum expected profit and the number of sausages that should be bought to achieve this.

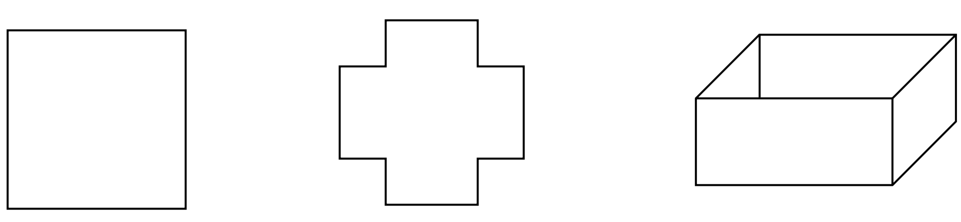
Q4 The relation between expected profit and number of sausages bought for a sizzle is $p = 0.8s - 0.003s^2$. Find the maximum profit and the number of sausages that should be bought to achieve this.

Q5 The concentration, c (in mg/L) of a drug in the bloodstream t hours after taking a 25 mg tablet is given by $c = 10te^{-0.8t}$. What is the maximum concentration and how long after taking the drug does the maximum occur? Check your answer by graphing with your calculator.

Finding the Objective Function

In the lemonade, sausage and drug problems in Q1 to Q5, you were given the objective function. In many optimisation problems you have to work it out yourself. Here is an example.

A tray is to be made from a 20 cm by 20 cm square of card by cutting a square from each corner, then folding up the sides.



If the squares are very small, the height (depth) of the tray will be very small and so the capacity will be very small. As the squares get bigger, the capacity will increase, but if they get too big, the base area of the tray will become very small and the capacity will decrease again.

Our task is to find the size square that will give the greatest capacity for the tray and to find that maximum capacity. This is how we do it.

The control quantity is the size of the squares. Let s be the side length of the squares.

The objective quantity is the capacity of the tray. Let V be the capacity of the tray.

We need to find the objective function, the formula for V in terms of x.

The base of the tray will be square with side lengths $20 – 2x$. The height of the tray will be x. So the volume will be $(20 – 2x)^2 \times x$ and our objective function is $V = (20 – 2x)^2 \times x$. Done!

Now, we go ahead with the calculus.

$$V = (20 - 2x)^2 \times x$$

$$= (400 - 80x + 4x^2) \times x$$

$$= 400x - 80x^2 + 4x^3$$

$$\frac{dV}{dx} = 400 - 160x + 12x^2$$

Then we find where the derivative is equal to zero

$$400 - 160x + 12x^2 = 0$$

$$100 - 40x + 3x^2 = 0$$

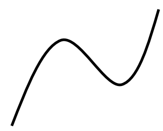
$$(3x - 10)(x - 10) = 0$$

$$x = 3.333 \quad \text{or} \quad x = 10$$

So the function has two stationary points – one at $x = 3.333$ and one at $x = 10$. However, it is not immediately obvious which is the maximum. We learnt a number of ways of determining the nature of a stationary point in Module C6-13 (Graph Sketching). They were looking at the sequence of stationary points, the y-values before and after, the gradient before and after, and the second derivative. As optimisation problems generally have a context, we can add one more to that list here – looking at the context. And we will use that approach here.

We have a stationary point when x is 10. But, when $x = 10$, the squares cut out of the corners of the sheet will take up the hole sheet, so the base area of the tray will be zero and the volume will be zero. Clearly this will be a minimum, not a maximum. So the maximum is at $x = 3.333$.

A similarly easy alternative would be to consider the sequence of stationary points. As the objective function is a cubic with positive leading term, the shape must be like this. Thus, the first stationary point will be a maximum and the second a minimum.



Now that we know that $x = 3.333$ gives the maximum capacity, we can find the maximum capacity by subbing 3.333 for x in the objective function. This gives:

$$V = (20 - 2x)^2 \times x$$
$$= (20 - 2 \times 3.333)^2 \times 3.333$$
$$= 593$$

So we get a maximum capacity of 593 cm^3 when 3.333 cm squares are cut from the corners.

Practice

Q6 A tray is to be made from a 40 cm by 40 cm square card by cutting a square from each corner, then folding up the sides. What size square will give the greatest capacity and what capacity will this produce?

Q7 A tray is to be made from a 20 cm by 30 cm card by cutting a square from each corner, then folding up the sides. What size squares will give the greatest capacity and what capacity will this produce?

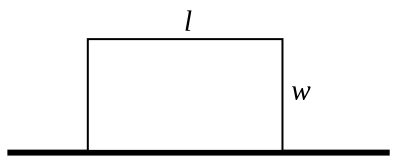
Q8 As a punishment for carving his name in the barmaid at the local pub, Paddy was allowed to choose any real number between 1 and 20. His number would then be cubed and the result of taking his number from 20 would be squared. The results would then be added. The resulting number was the number of days for which he would be banned from the pub. What numbers should he choose (assuming he wants to be banned for as little time as possible)? How long would he be banned?

Q9 A lifesaver standing on the edge of a lake. A drowning swimmer is 60 m from the water's edge and 200 m from the lifesaver. If the life saver can run at 5 m/s and swim at 1 m/s, what is the minimum amount of time she will need to reach the swimmer? [Assume that the water's edge is straight.]

Two Independent Variables in the Objective Function

In the tray examples above, the objective function had one independent variable. It was x, the size of the squares. In many optimisation problems, however, the objective function will have two independent variables. Here is an example.

Sandy wants to make a rectangular enclosure for her ferret. She has 20 m of fencing, which she will use for three of the sides. For the fourth side, she will use an existing wall.



What is the largest area she can make and what dimensions will give her this area?

Here we want to optimise *area*, so *area* is the objective quantity. As the enclosure is a rectangle, our objective function is $A = l \times w$, where A is the area in square metres, l is the length in metres and w is the width in metres.

But A is a function of l and w, so there are two independent variables.

Whenever there are two independent variables, the value of a **constraint quantity** will be given. This will be the numerical information provided in the problem. In the case of Sandy's ferret enclosure, the constraint quantity is the length of fencing and its value is 20 m.

The constraint quantity will be a function of both the independent variables. What we have to do is write the formula for the constraint quantity. If the length of fencing is f, then the formula is

$$f = l + 2w.$$

Then we substitute the given value for f to get

$$20 = l + 2w$$

Then we rearrange this to make one of the independent variables the subject:

$$l = 20 - 2w$$

Then we sub this into the objective function:

$$A = l \times w$$

$$A = (20 - 2w) \times w$$

$$A = 20w - 2w^2$$

Now we have our objective function with just the one control quantity and we can proceed as before – differentiate and find where the derivative is equal to zero.

$$A = 20w - 2w^2$$

$$\frac{dA}{dw} = 20 - 4w = 0$$

$$20 = 4w$$

$$w = 5$$

Now we answer the question, which was: find what dimensions will give her the largest area and find the largest area.

We have $w = 5$. We have a relation between l and w: $l = 20 - 2w$. From this we get $l = 10$.

And we have $A = l \times w$. Subbing into this we get $A = 10 \times 5 = 50\text{ m}^2$.

So the largest area is 50 m^2 and the dimensions that produce this are 10 m by 5 m.

There are four steps to solving this type of problem. They are FODA:

F Write a **FORMULA** for the quantity that we wish to optimise (write the objective function). This will often have a couple of independent variables, e.g. l and w.

O Write a formula for the constraint quantity. This should have the same independent variables. Then rearrange it to make one of the independent variables the subject. Then sub into the objective function from Step **F** to get a formula with just **ONE** independent variable.

D **DIFFERENTIATE** the objective function, put the derivative equal to zero, and solve.

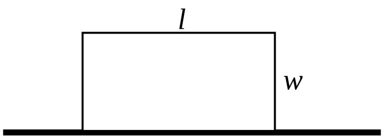
A **ANSWER** the question.

Don't forget the last step. In some questions, you will be asked for the values of the independent variable; in others, you will be asked for the value of the objective quantity; in yet others, you will be asked for both. Before doing step A, it is worth going back and re-reading the question.

Practice

Apply the FODA approach to the following questions.

Q10 Cowboy Joe wants to make a pen for his cow. He has 50 m of fence and plans to make a rectangular pen using his 50 m for three sides of the rectangle and an existing wall for the other side. Find the greatest possible area and the dimensions which will give that area.



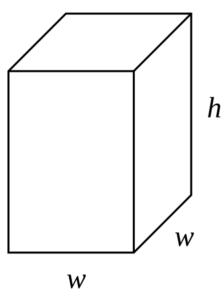
Q11 Find the maximum area for Joe's pen if he had 60 m of fencing.

Q12 What would be the largest area for Joe's pen if he had 50 m of fencing plus a 2 m wide gate hung from the wall?

Q13 Cindy has 32 m of fence to make a rectangular pen for her pet weasel. She doesn't have a wall, so needs to make all four sides out of her fencing. What is the largest possible area and what dimensions will produce this area?

Q14 A cardboard box is to be made in the shape of a square-based prism with no lid. It is to have a volume of 1000 cm^3. We need to work out the minimum area of cardboard that could be used.

(a) We need to optimise the area of cardboard, so call it A. Then write a formula for the A in terms of the width, w, and height, h, of the box.



(b) We are told the volume of the box, so write a formula for the volume in terms of w and h.

(c) Sub 1000 cm^3 for h.

(d) Rearrange the volume formula to make h the subject.

(e) Substitute the formula for h into the area formula so that we have a formula for area in terms if just one independent variable, w.

(f) Differentiate and find the value of w that gives the minimum area.

(g) Find the minimum area.

Q15 A cardboard box is to be made in the shape of a square-based prism with no lid. It is to have a volume of 600 cm^3. What dimensions will use the smallest area of cardboard?

Q16 A cardboard box is to be made in the shape of a square-based prism with no lid. If 0.2 m^2 of cardboard is to be used, find the maximum capacity of the box in litres.

Optimisation takes a lot of practice. Here is some more.

Practice

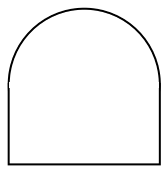
Solve these problems using calculus.

Q17 A pet food company is designing cylindrical dog meat cans to hold 800 mL. What dimensions will require the smallest area of metal?

Q18 A box is to be made out of card in the shape of a square–based prism. It is to be made of a single layer of card, except for the lid, which will be made of two layers. Find the minimum area of card required if the capacity is to be 500 cm^3.

Q19 A box is to be made out of card in the shape of a square–based prism. The sides will be made of a single layer of card, but the base and lid will be made of two layers. Find the maximum capacity of the box if a total of 0.5 m^2 of card can be used.

Q20 A window is to be made in the shape of a rectangle topped by a semi-circle. If the perimeter cannot exceed 4 m, what is the largest possible area?



Q21 Find the volume of the largest cylinder that will fit inside a sphere of radius 1 m.

Solve

Q51 A sector of a circle has area 20 cm^2. What radius will allow it to have the smallest perimeter?

Q52 $y = x^3 - 4x$. Without your calculator, determine the highest and lowest values of y when $0 \le x \le 3$.

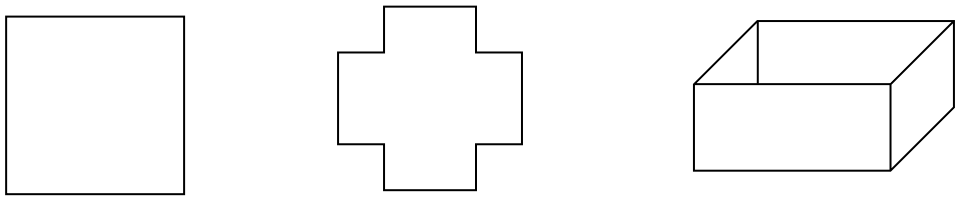
Q53 Find the shortest distance between the point (4, 8) and the parabola $y = x^2$.

Revise

Revision Set 1

Q61 The expected return, r, on an investment, x (both in dollars) is given by $r = 1.5x - 0.0004x^2$. What investment will give the highest expected return and what will the expected return be for that investment?

Q62 A tray is to be made from a 30 cm by 30 cm square of card by cutting a square from each corner, then folding up the sides.



What is the greatest possible capacity for the tray? How big will the squares have to be to produce this capacity?

Q63 A cardboard box is to be made in the shape of a square-based prism with no lid. It is to have a volume of 1 L. Find the minimum area of cardboard that could be used.

Answers

Q1 40 L   Q2 25.8 L, \$172.13   Q3 \$50, 100 sausages   Q4 \$53.33, 133 sausages
Q5 4.6 mg/L, 1.25 hours   Q6 6.667 cm, 4741 cm^3 Q7 3.92 cm, 1056 cm^3
Q8 $3^{1}/_{3}$, 314.8 days   Q9 96.95 seconds
Q10 312.5 m^2, 25 m by 12.5 m   Q11 450 m^2 Q12 338 m^2 Q13 64 m2, 8 m by 8 m
Q14 (a) $A = w^2 + 4wh$ (b) $V = w^2h$ (c) $h = \frac{V}{w^2}$ (d) $h = \frac{1000}{w^2}$ (e) $A = w^2 + \frac{4000}{w}$
(f) $\frac{dA}{dw} = 2w - \frac{4000}{w^2} = 0$, $w = 12.6$ cm (g) 357 cm^2
Q15 width = 9.76 cm, height = 4.88 cm   Q16 6.16 L
Q17 radius = 5.03 cm, height = 10.06 cm   Q18 432.7 cm^2 Q19 17 L
Q20 1.12 m^2 Q21 2.418 m^3

Q51 4.47 cm   Q52 15, −3.079   Q53 1.154

Q61 \$1875 Q62 2 L, 5 cm Q63 476 cm^2