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C6-14 Graph Sketching

- sketching graphs of polynomial and rational functions using knowledge of function shape, extreme behaviour, discontinuities, axis intercepts, stationary points and spot values

Summary Learn Solve Revise Answers

Summary

To sketch the graph of a polynomial function we collect data on the function and put it onto a set of axes until we have enough to tie down the shape sufficiently. The data comes from consideration of:

- the shape of polynomial functions of that degree
- extreme behaviour (what happens to y as x becomes very large, positive and negative)
- the y-intercept and, if easy to find, any x-intercepts
- stationary points (if easy to find)

Any uncertainties can then be filled in using spot values (y-values for chosen x-values).

The same methods can be used with rational functions, along with consideration of discontinuities (gaps and asymptotes).

Learn

The tradition of learning to sketch graphs dates from before the days of graphics calculators. Nowadays, any function can be graphed accurately and instantly with a calculator, so one might be excused for considering this skill obsolete.

However, it is still part of many maths curricula. The rationale for learning to sketch graphs is that it develops some worthwhile concepts. Some of these help develop a feel for the shapes of graphs and the reasons why graphs have certain features. Also, the process of graph sketching involves practice of algebra and calculus skills.

When sketching a graph, we draw a set of axes, then gradually add information as we find it. Finally, we join it all up to produce the sketch.

Polynomial Functions

Below are step-by-step instructions for sketching graphs of polynomial functions. The example $y = 2x^3 - 3x^2 - 12x + 5$ is used to illustrate.

Function Shape

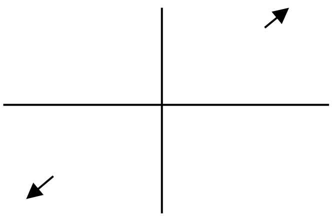
You were introduced to the shape of polynomial functions in Module A5-1. A polynomial of degree n has up to n arms with stationary points between them.

Our example $y = 2x^3 - 3x^2 - 12x + 5$ has at most 3 arms and two stationary points.

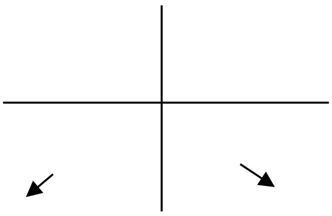
Extreme Behaviour

Extreme behaviour refers to what happens to the dependent variable (we will call it y in this module) as the independent variable (we will call it x) becomes very large positive and very large negative. In both cases, y will become very large, either positive of negative. To determine which, just sub a large positive number for x in the term with the highest power and then do the same with a large negative number.

In our example function $y = 2x^3 - 3x^2 - 12x + 5$, $2x^3$ is the term with the highest power of x. When x is large and positive, $2x^3$ will be large and positive and so y will be large and positive; when x is large and negative, $2x^3$ will be large and negative and so y will be large and negative. We can put this information onto the graph like this.



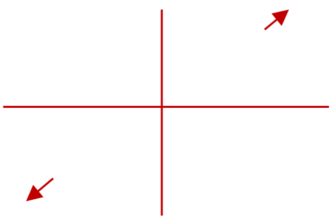
With the function $y = 4 + 3x^2 - x^4$, however, when x is large and positive, $-x^4$ and therefore y will be large and negative, and when x is large and negative, $-x^4$ and y will be large and negative. So we would put this information on the graph like this.



The reason we only need to look at the highest-power term is that, as x becomes very large, this term will always outweigh all the others.

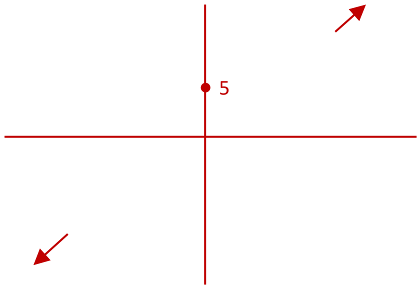
There is a shortcut to this process: knowing that even-degree polynomials with positive leading terms are $\cup$ shaped, with negative leading terms are $\cap$ shaped; and that odd-degree polynomials with positive leading terms are rising s-shapes, with negative leading terms are falling s-shapes.

For our example function $y = 2x^3 - 3x^2 - 12x + 5$, we have



Axis Intercepts

The y-axis intercept is always easy to get. We just sub 0 for x in the formula and find y. Once we find it, we mark it on the graph with a dot on the axis and the number beside it.

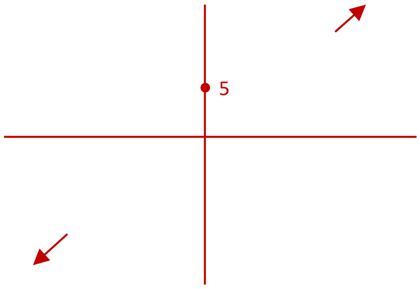


x-intercepts are easy to get if the formula is in factorised form. For example, $y = (x + 3)(x - 2)(2x + 1)$ has x-intercepts of −3, 2 and ½.

If the formula is not factorised, but is easy to factorise, e.g. $y = x^3 - 5x^2 + 6x$, then we should factorise it, in this case to $x(x - 2)(x - 3)$, to see that the intercepts are 0, 2 and 3. If it is not easy to factorise, then we generally don't bother with the x intercepts.

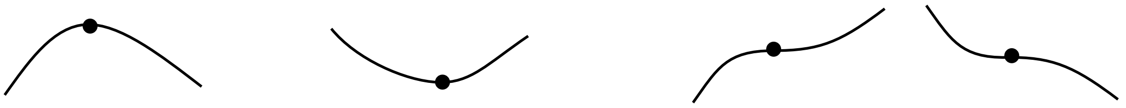
If we do get axis intercepts, we mark them on the graph with dots and numbers.

Our example function, $y = 2x^3 - 3x^2 - 12x + 5$, has a y-intercept of 5. It is difficult to factorise, so we won't get any x-intercepts.



Stationary Points

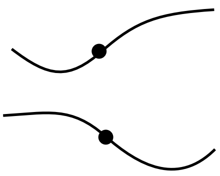
A stationary point is a horizontal point on the graph where the gradient is 0, i.e. the derivative is zero. It may be a maximum, a minimum or a point of horizontal inflection.



maximum　　minimum　　points of horizontal inflection

Stationary points are so called because the y-value is neither rising nor falling, but stationary. Maxima and minima are also called *turning points*.

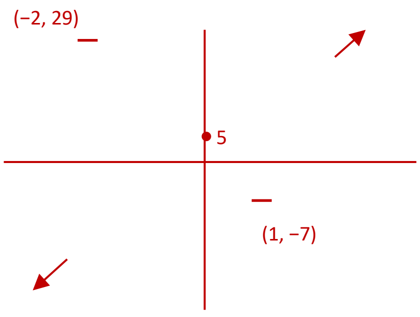
Note that not all points of inflection are horizontal: a point of inflection is a point where the curve changes from curving up to curving down or vice versa. However, we only need be concerned with points of horizontal inflection.



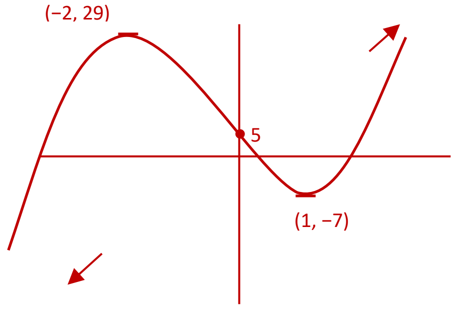
Polynomials are easy to differentiate, so we differentiate, equate the derivative to zero, then solve the equation – if it is reasonably easy to solve.

This gives us the x-coordinates of the stationary points. We get the y-coordinates by subbing the x-value into the original formula (not the derivative). We then mark the stationary points on the graph with short horizontal lines and their coordinates.

The derivative of our example is $\frac{dy}{dx} = 6x^2 - 6x - 12$. This factorises easily to $6(x - 2)(x + 1)$. Equating this to 0, gives us $x = -2$ and $x = 1$. The y-values are 29 and -7 respectively. So we now have:



We may have enough information now to sketch the graph. In the case of our example, we do.



Determining the Nature of Stationary Points

In other situations, we may need more information to tell whether the stationary points are maxima, minima or points of horizontal inflection. There are four ways of identifying the nature of the stationary points – sequence, y-values before and after, gradient before and after, second derivative. We use the one that will be easiest.

Sequence

The gradient of the graph alternates between positive and negative between the stationary points. Having determined the extreme behaviour, we will know whether the graph rises from large negative y-values or falls from large positive y-values as we come in from the left. Let's say it rises, as $y = x^5 - 4x^3 + 2x$ would rise. The first turning point must then be a maximum, the next a minimum, the next a maximum and so on.

This is unless there is a point of horizontal inflection. A point of horizontal inflection is indicated when we solve the equation 'derivative = 0'. In doing this, we will use the null factor theorem with a series of factors to get a series of solutions. If any of the solutions occur twice, as in $(x + 1)(x + 2)(x + 2) = 0$, then that will indicate that the gradient crosses from negative to positive, and immediately crosses from positive back to negative again. This indicates a point of horizontal inflection rather than a maximum or minimum.

If a solution occurs 3 times, the gradient will change sign 3 times on the same spot and the result will be a maximum or minimum. 4 times indicated a point of horizontal inflection, and so on.

y-values Before and After

The second method for determining the nature of a stationary point is to find the y-values just before and just after the point. For instance, if there are stationary points at (1, 4) and (6, −2), we can find the nature of the first by finding y when $x = 0$ and when $x = 2$. If both are less than 4, then it is a maximum, and so on. Then do similar for the second, unless you then have enough information to know without testing it.

Gradient Before and After

It may be easier to find the value of the derivative before and after the turning point. If it is >0 before and <0 after, then we have a maximum, and so on.

Second Derivative

If the second derivative at the turning point is positive, then it is a minimum; if negative, it is a maximum; if zero, then it is a point of horizontal inflection.

Spot Values

By the time you have considered the function shape, the extreme behaviour, the axis intercepts and the stationary points, you will often have enough information to confidently sketch the graph.

But sometimes, you may not, particularly if you haven't found the x-intercepts or stationary points because the equations are too hard to solve.

If you haven't got enough information, you need to find a few points on the graph by picking the x-value, then subbing that into the formula to get the y-value. Pick points that will best tie down where the graph goes and mark the points on your graph with a dot and the coordinates as you get them.

Practice

Q1 Use the methods detailed above to sketch the graphs of the following functions. You may look through the instructions as you go for the first ones, but try to do the later ones without looking. Use your calculator to check your sketches.

(a) $y = (x - 2)(x + 1)(x + 3)$

(b) $y = x^3 - 5x^2 + 6x$

(c) $y = x^4 - x^2$

(d) $y = -x^3 - 3x^2 + 24x + 3$

Q2 Find a polynomial function with x-intercepts at -3, -1, 2 and 5 and a y-intercept of 3. Use your calculator to check.

Rational Functions

In graphing rational functions, we can use some of the same techniques that we learnt for polynomials, but we also have to consider discontinuities (asymptotes and gaps). Regarding discontinuities, it could be worth having another look at the sections on discontinuities and rational functions in Module A5-10 (Further Relations) before going on.

A rational function can be written as $y = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. An example is $y = \frac{3x^3-2x+21}{x^2+2x}$.

Function Shape

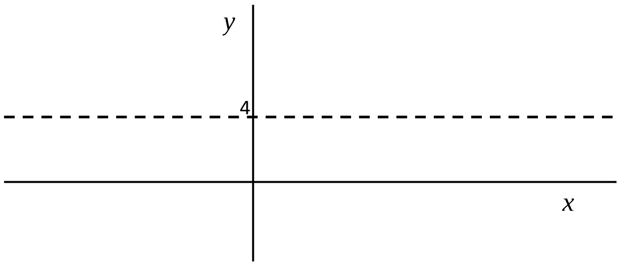
The shapes of rational functions are harder to predict than those of polynomials. For this reason, we normally skip this step.

Extreme Behaviour

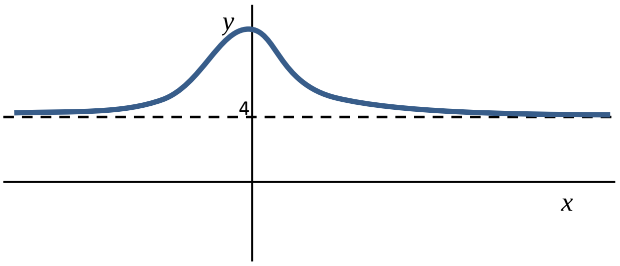
We divide the leading term of $P(x)$ by the leading term of $Q(x)$ (when both are in expanded form). Call the result $R(x)$. $R(x)$ will be of the form ax^n, where n is an integer.

If $n > 0$, then the extreme behaviour of the rational expression will be the same as for a polynomial with ax^n as the leading term.

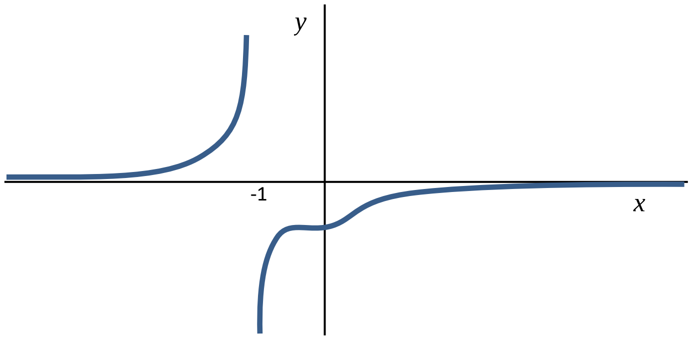
If $n = 0$, then, as x becomes large, positive of negative, y approaches a. We say that there is a horizontal asymptote at $y = a$. We draw a dotted line to show the position of the asymptote. For example, for the rational function $y = \frac{4x^2-2x+21}{x^2+2}$, $R(x) = 4$, so y approaches 4 as x becomes very large, positive or negative. We draw a dotted line for the asymptote at $y = 4$.



The actual graph looks something like this, though we haven't got t that stage yet.

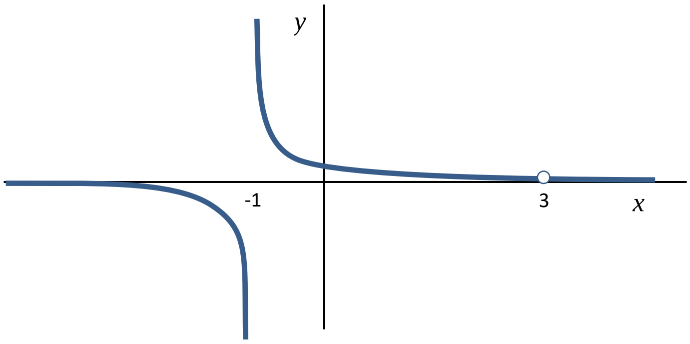


If it has a negative power of x, (e.g. x^{-1} or x^{-4}) then y will approach zero as x gets large, both positive and negative. We say that there is a horizontal asymptote at $y = 0$. The graph of $y = \frac{x^2-5}{x^3+1}$ looks something like this:



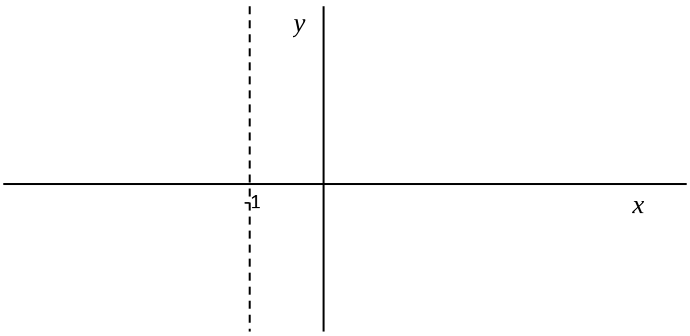
Discontinuities

If, when factorised, there is a factor $(x - a)$ on the bottom and on the top, then the function will be undefined at $x = a$, so there will be a gap discontinuity at $x = a$. For example, $y = \frac{x-3}{(x-3)(x+1)}$ will look like this:



If there is a factor $(x - b)$ on the bottom, but not on the top, then the function will be undefined at $x = b$ and the function will approach ∞ or $-\infty$ as x approaches b and so there will be a vertical asymptote at $x = b$. In the graph above, there is a factor $(x + 1)$ on the bottom, but not on the top, so there is a vertical asymptote at $x = -1$.

When sketching the graph, we may not yet know whether y is positive or negative to the left and right of the asymptote, so we just draw in a vertical line at the x-value of the asymptote, like this:



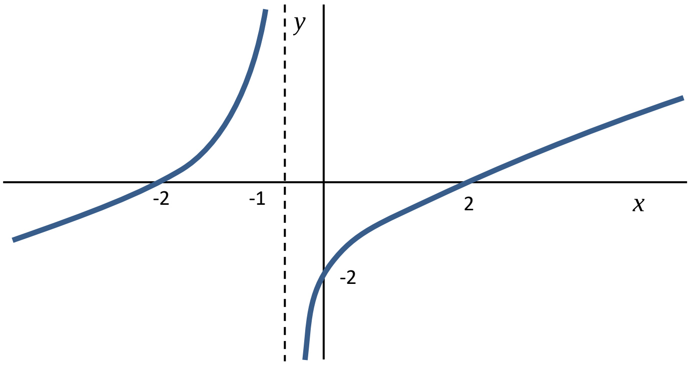
There may be more than one vertical asymptote if there is more than one factor on the bottom which doesn't occur on the top.

Axis Intercepts

To get the y-intercept, just sub $x = 0$ into the formula.

The x-intercepts will occur where $y = 0$. This is where the top of the rational expression, $P(x)$ equals 0. If this is easy to solve, solve it. Otherwise, don't bother.

For example, the y-intercept of $y = \frac{x^2-4}{3x+2}$ will be -2 and the x-intercepts will be -2 and 2.



Stationary Points

Differentiating a rational function often leads to a much more complicated rational function and equating that function to zero can produce an equation which is hard to solve. If using stationary points is easy, go ahead, but if not, don't bother.

Spot Values

We often have to resort to these because of lack of information from the other methods. Once you have all the information from other sources that can be obtained easily, then use spot values to find what the curve does in between the known points.

In particular, finding the y-value either side of a vertical asymptote may be necessary to find whether the graph goes to ∞ or $-\infty$.

Practice

Q3 Use the methods detailed above to sketch the graphs of the following rational functions. Use your calculator to check your sketches.

(a) $y = \frac{(x+2)(x-3)}{x+1}$

(b) $y = \frac{3(x+3)(x-1)}{x(x-1)}$

(c) $y = \frac{(x+2)}{x^2+x}$

(d) $y = \frac{2x^2+4x-6}{x^2+3x}$

Q4 Find a rational function with x-intercepts of -3 and 4 and asymptotes at $x = 2$ and at $y = -2$. Use your calculator to check.

Solve

Q51 Graph $y = x^x$. Do as much as you can before using your calculator to check.
Hint 1: use a window $-2 \le x \le 2$, $-2 \le y \le 2$.
Hint 2: as this is not a polynomial or rational function, most of the methods for those functions won't help: you will need to get most of your information from spot values.
Hint 3: you will need the index laws.
Hint 4: there should be points in three of the four quadrants, but not on either of the axes.
Hint 5: When you check your result, your calculator may need to use a fairly small view window to show all the details and, even, then, it may show lines where there shouldn't be any.
Discuss any unusual features of the graph.

Revise

Revision Set 1

Q71 Sketch the graphs of the following functions. Use your calculator to check your sketches.

(a) $y = (1 - x)(x + 1)(x + 4)$

(b) $y = x^3 + 5x^2 + 6x$

(c) $y = x^3 - 9x^2 + 27x + 5$

(d) $y = \frac{(x+5)(x-2)}{x+1}$

(e) $y = \frac{x^2+x-2}{x^2-x}$

Q72 Find a polynomial function with x-intercepts at −3, −1, 2 and 5 and a y-intercept of 3. Use your calculator to check.

Q73 Find a rational function with x-intercepts of −3 and 4 and asymptotes at $x = 2$ and at $y = -2$. Use your calculator to check.

Answers

All answers can be checked using a graphics calculator.