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C6-13 Higher-order Derivatives

- second and higher-order derivatives

Summary Learn Solve Revise Answers

Summary

The second derivative of a function is the derivative of the derivative of that function.

The second derivative of y with respect to x is written $\frac{d^2y}{dx^2}$ and pronounced *d 2 y by d x squared*.

The second derivative of displacement is acceleration. The second derivative of any function is related to the curvature of the graph of that function.

Differential equations with second derivatives are called second-order differential equations and require two boundary conditions to get a particular solution.

Higher order derivatives are written as $\frac{d^3y}{dx^3}$ etc.

Learn

Second Derivatives

The second derivative of a function is the derivative of the derivative of the function.

Suppose we have a formula for displacement $s = 100 + 40t - 5t^2$.

The rate of change of s with respect to t gives us the velocity. So $v = \frac{ds}{dt} = 40 - 10t$.

The rate of change of v with respect to t gives us the acceleration. So $a = \frac{dv}{dt} = -10$.

a is the derivative of the derivative of s. We call it the second derivative of s. If we need to distinguish it, we call v the first derivative of s.

As we saw in Module C6-3, we can write $\frac{ds}{dt}$ as $\frac{d}{dt}s$, pronounced *d by dt of s*.

Similarly, the second derivative of s can be written $\frac{d}{dt}\frac{d}{dt}s$. However, we generally shorten this (with slight poetic licence) to $\frac{d^2s}{dt^2}$ and this is the usual notation for the second derivative of s with respect to t. It is pronounced *d 2 s by d t squared*.

When using function notation, if $y = f(x)$, the first derivative is $y' = f'(x)$ and the second derivative is $y'' = f''(x)$.

Practice

Q1 Find $\frac{d^2s}{dt^2}$ if

(a) $s = 3t^2$ (b) $s = 4t - 5t^3$ (c) $s = \sin t$ (d) $s = e^t$

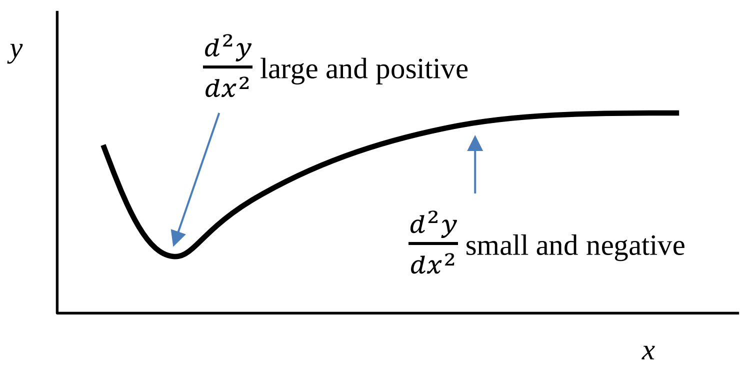
Q2 Find $\frac{d^2y}{dx^2}$ if

(a) $y = \ln x$ (b) $y = (4x + 2)^5$ (c) $y = xe^x$ (d) $y = \cos x^2$

Physical Meaning

The second derivative of displacement is acceleration. For other functions, the second derivative may not have such a familiar meaning. For instance, the area of a circle is $A = \pi r^2$. The first derivative is $\frac{dA}{dr} = 2\pi r$. This is the rate at which the area changes as r increases, i.e. the increase in area for a small unit increase in r. It is the circumference. The second derivative is $\frac{d^2A}{dr^2} = 2\pi$. This is the rate at which the circumference changes as r increases, i.e. the increase in circumference for a small unit increase in r. $\frac{d^2A}{dr^2}$ is not as useful a concept as acceleration.

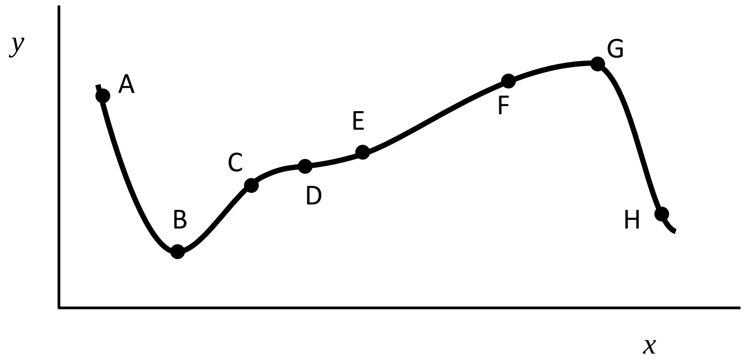
If we graph y as a function of x, however, $\frac{dy}{dx}$ is the gradient of the curve at any value of x and $\frac{d^2y}{dx^2}$ is the rate at which the gradient is changing, i.e. the rate at which the graph is curving: a large value for $\frac{d^2y}{dx^2}$ means a sharp curve at that x-value; a small value for $\frac{d^2y}{dx^2}$ means a gentle curve. A positive value means the graph is curving upwards (gradient increasing); a negative value means the graph is curving



downwards (gradient decreasing).

Practice

Q3 For each of points A to H on the graph below, say whether $\frac{d^2y}{dx^2}$ is large or small, positive, negative or zero.



Solving Second-Order Differential Equations

A second-order differential equation is one that contains a second derivative. The differential equations we have met to date have been first-order.

Suppose we know that the gravitational acceleration of a projectile is −9.8 m/s/s upwards and we need to calculate the height of the projectile at a certain time. We have $\frac{d^2h}{dt^2} = -9.8$. To get a formula for h we have to undo the two differentiations by anti-differentiating twice. Thus we will need two boundary conditions, one for each of the constants of integration. The ones we are most likely to have are the initial (when $t = 0$) height and the initial velocity. Let's say the initial height is 50 m and the initial velocity is 40 m/s (upwards).

$$\frac{d^2h}{dt^2} = -9.8$$

$$\frac{dh}{dt} = -9.8t + c$$

When $t = 0$, $\frac{dh}{dt} = 40$ $\therefore$ $40 = c$ and $\frac{dh}{dt} = -9.8t + 40$

$$h = -4.9t^2 + 40t + c$$

When $t = 0$, $h = 50$ $\therefore$ $50 = c$ and $h = -4.9t^2 + 40t + 50$

Practice

Q4 The acceleration of a projectile travelling vertically under gravity is −9.8 m/s/s. When $t = 0$, $h = 20$ m and $v = 75$ m/s. Find the relation between h and t and the height when $t = 10$.

Q5 Solve $\frac{d^2y}{dx^2} = 4x$, given that $x = 5$ and $\frac{dy}{dx} = 2$ when $t = 1$.

Higher-Order Derivatives

The derivative of the second derivative is the third derivative, $\frac{d^3y}{dx^3}$ or $f'''(x)$, and so on.

These don't often have useful meaning, but are used in some situations in higher mathematics like Maclaurin's expansions. You probably don't need to know this at this stage, but McLaurin's expansion allows us to express any function as a polynomial of infinite degree. It uses the fact that:

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \qquad \text{i.e. } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n,$$

This allows us to write $\cos x$ as $\frac{\cos 0}{0!} + \frac{-\sin 0}{1!}x + \frac{-\cos 0}{2!}x^2 + \frac{\sin 0}{3!}x^3 + \frac{\cos 0}{4!}x^4 + \ldots$

$$= \frac{1}{0!} + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots$$

This is how calculators work out cosines.

Practice

Q6 Find (a) $\frac{d^4}{dx^4}(x^6 - 4x^2)$ (b) $\frac{d^{33}}{dx^{33}}\sin x$

Q7 Find the Maclaurin's expansions of (a) $\sin x$ (b) e^x (c) $\ln x$

Solve

Q51 Find the nth derivative of xe^x

Q52 If i is the imaginary number $\sqrt{-1}$ such that $i^2 = -1$, use Maclaurin's expansions to prove that $e^{ix} = \cos x + i \sin x$.

Q53 Use the result from Q51 to show that $e^{i\pi} + 1 = 0$. This is known as Euler's Identity and is the most loved equation among mathematicians, partly because it contains the five most fundamental numbers of mathematics (0, 1, i, e and π)

and almost nothing else.

Q54 Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \cos x$. [Guess and check is probably the best approach.]

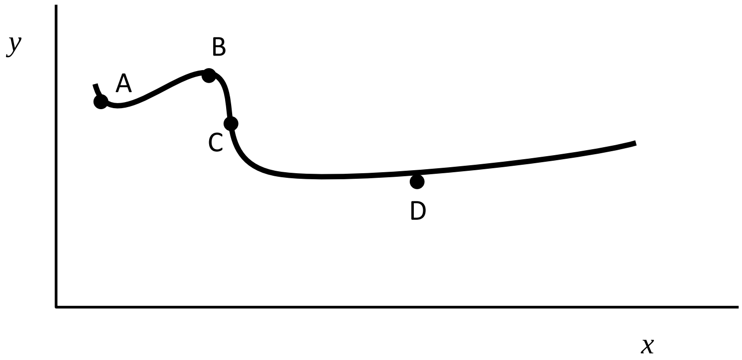
Revise

Revision Set 1

Q61 Find $\frac{d^2y}{dx^2}$ if

(a) $y = x^3$ (b) $y = (2x + 5)^4$ (c) $y = x \ln x$ (d) $y = \cos x$

Q62 For each of points A to D on the graph below, say whether $\frac{d^2y}{dx^2}$ is large or small, positive, negative or zero.



Q63 If the upward acceleration of a rocket is given by $a = 0.6t$, where t is in seconds and a is in m/s/s, and the rocket was at height 200 m and moving upwards at 30 m/s when $t = 6$,

(a) find the formula for height, h.

(b) find the height when $t = 20$.

Q64 Find (a) $\frac{d^3}{dx^3}(x^6 + 3x^2)$ (b) $\frac{d^3}{dx^3}\ln x$

Answers

Q1 (a) 6 (b) $30t$ (c) $-\sin t$ (d) e^t

Q2 (a) $y = \frac{-1}{x^2}$ (b) $y = 80(4x + 2)^3$ (c) $y = xe^x + 2e^x$ (d) $y = -2x\cos x - 2\sin x$

Q3 A small positive B large positive C small negative D zero
E small positive F small negative G large negative H small positive

Q4 (a) $h = -4.9t^2 + 55.4t + 8.8$ (b) 72.8 m

Q5 $y = \frac{2}{3}x^3 + 4\frac{1}{3}$

Q6 (a) $360x^2$ (b) $\sin x$

Q7 (a) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots$ (b) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$

Q51 $xe^x + ne^x$ Q54 $y = \sin x$

Q61	(a) $6x$	(b) $48(2x+5)^2$	(c) $\frac{1}{x}$	(d) $-\cos x$
Q62	A large positive	B large negative	C zero	D small positive
Q63	(a) $h = 0.1t^3 + 19.2t + 63.2$		(b) 1247.2 m	
Q64	(a) $120x^3$		(b) $\frac{2}{x^3}$	