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C6-12 Pert

- differential equations of the form $\frac{dA}{dt} = rA$

Summary Learn Solve Revise Answers

Summary

The solution of a differential equation of the form $\frac{dA}{dt} = rA$ is $A = Pe^{rt}$ where P is the initial value, i.e. the value of A when $t = 0$.

We can use this solution to solve problems. The process involves three steps:

1. write the general formula, $A = Pe^{rt}$
2. use the given information to find the values of the parameters r and P
3. use the particular formula, e.g. $A = 200e^{0.2t}$ to find a value of A or t.

Learn

Differential equations of the form $\frac{dA}{dt} = rA$

In all the differential equations we have dealt with so far, we were given a differential equation in which the derivative was equal to a function of the independent variable.

Here, we will look at equations of the form $\frac{dA}{dt} = rA$, where r is a constant. In these, the derivative is a function of the dependent variable, A.

$\frac{dA}{dt} = rA$ states that the rate of change of a quantity is proportional to the value of that quantity. This is a characteristic of an exponential relation. So the solution is an exponential function. It is $A = Pe^{rt}$, where P is the initial value of A, i.e. the value of A when $t = 0$.

The solution of $\frac{dA}{dt} = rA$ is $A = Pe^{rt}$ where P is the value of A when $t = 0$.

We commonly use this solution to solve problems without working through its derivation as in the following example. There are three steps to solving problems this way:

1. write the general formula, $A = Pe^{rt}$
2. use the given information to find the values of the parameters r and P
3. use the particular formula, e.g. $A = 200e^{0.2t}$ to find a value of A or t.

Example 1

A bacterial colony is growing at a rate in mg/h equal to 0.2 times the amount already present in mg. If there is 400 mg at 2 p.m., how much will there be at 9 p.m. and when will there be 5 g?

Let A be the amount present in mg and t be the time in hours since midday

$$\frac{dA}{dt} = rA$$

$$A = Pe^{rt}$$

$r = 0.2$, $\quad \therefore A = Pe^{0.2t}$

When $t = 2$, $A = 400$. So $400 = Pe^{0.4}$ and $P = 400 \div e^{0.4} = 268$

$$\therefore \mathbf{A = 268e^{0.2t}}$$

When $t = 9$, $\quad A = 268e^{1.8} = 1621$

So at 9 p.m. there will be 1621 mg.

To find when there will be 5 g, we sub 5000 for A to get

$$5000 = 268e^{0.2t}$$

$$18.66 = e^{0.2t}$$

$$\ln 18.66 = 0.2t$$

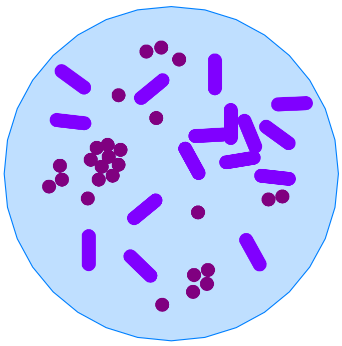
$$2.93 = 0.2t$$

$$t = 14.63$$

$$= 14 \text{ h } 38 \text{ min}$$

So there will be 5 g at 2:38 a.m. that night.

That solution is fine, though it would have slightly quicker to let t be the time since 2 p.m.
Then P would have been 400.



Example 2

A fungus grows at a rate proportional to the amount present. If there is 30 mg at 9 a.m. and 46 mg at 7 p.m. the same day, how much will there be at 9 a.m. the next day?

Let A be the amount present in mg and t be the time in hours since 9 a.m.

$$\frac{dA}{dt} = rA$$

$$A = Pe^{rt}$$

When $t = 0$, $A = 30$. So $P = 30$

$$\therefore\ A = 30e^{rt}$$

When $t = 10$, $A = 46$, so $46 = 30e^{10r}$

$$1.533 = e^{10r}$$

$$\ln 1.533 = 10r$$

$$0.427 = 10r$$

$$r = 0.0427$$

So $\boldsymbol{A = 30e^{0.0427t}}$

At 9 a.m. the next day, $t = 24$, so $A = 30e^{0.0427\times24}$

$$= 83.6 \text{ mg}$$

Practice

Q1 A bacterial colony is growing at a rate in mg/h equal to 0.5 times the amount in mg already present. If there is 500 mg at $t = 0$, how much will there be at $t = 5$?

Q2 A different bacterial colony is growing at a rate in g/h equal to 0.1 times the amount in mg already present. If there is 0.8 g at 4 p.m.,
(a) how much will there be at 11 p.m?
(b) when will there be 2 g?

Q3 A deposit account gets 6% p.a. interest compounding hourly. If the initial deposit is $5000, how much will be in the account after 4 years?
[You may consider this to be continuous growth.]

Q4 A fungal colony grows at a rate proportional to the amount present. There is 19 mg at 10 a.m. and 32 mg at 6 p.m. the same day. How much will there be at 8.a.m the next day and when will the mass reach 200 mg?

Q5 When an object is cooling, the rate at which its temperature drops is proportional to the temperature difference between the object and its surroundings. A cup of coffee is initially 60° above room temperature. Its temperature falls at a rate (in degrees per minute) of 0.08 times the temperature difference between the coffee and the room.

(a) What will be the temperature difference be 10 minutes later?

(b) How long will it take for the temperature difference to reach 10°?

Q6 Another cup of coffee is initially at 70°. The room temperature is 20°. Its temperature falls at a rate (in degrees per minute) of 0.1 times the temperature difference between the coffee and the room.

(a) What will be the temperature of the coffee after 15 minutes?

(b) How long will it take for the temperature of the coffee to fall to 25°?

Q7 A cup of tea is 50° above room temperature at 2:30 p.m. and 30° above at 2:45. p.m. How much above room temperature will it be at 3:10 p.m.?

Q8 The rate of decay of radioactive substances is always proportional to the amount present. The rate of decay of a sample of radioactive caesium in mg/h is 0.02 times the amount of sample remaining in milligrams. If there is 200 mg present 40 h after the sample was produced, how much would there have been initially?

Q9 When a radioactive sample decays, the half-life is the time required for half of the initial sample to decay. Carbon-14 has a half-life of 5730 years.

(a) How much of a 20 g sample of Carbon-14 would remain after 2000 years?

(b) How long would it take for 90% of a Carbon-14 sample to decay?

Q10 A tank has a leak in the base. The rate at which water leaks out is proportional to the amount of water in the tank. When the tank contains 500 L, water leaks out at 2 mL/s.

(a) If there is 400 L in the tank when it is first observed, how much will be left 3 days later?

(b) How long will it take for the amount of water in the tank to fall from 200 L to 50 L?

(c) How long will it take to go from 50L to empty?

Q11 A swimming pool contains 20 g of dirt suspended in 50 000 L of water. The pump filters 5000 L of water per hour, removing all the dirt in the process. What is the half life of the dirt? In other words, how long will it take for half the dirt to be removed?
[Note, the answer is not 5 hours. Assume that the pool water if being

continually mixed. This will mean that the water going through the filter will be getting gradually cleaner and so the rate of removal will decrease progressively.]

To solve $\frac{dA}{dt} = rA$ and get $A = Pe^{rt}$, we use a process called '*separating the variables*'. The two variables in the relation are A and t. Separating the variables means rearranging the equation $\frac{dA}{dt} = rA$ so that all the As are	$\frac{dA}{dt} = rA$
on one side and all the ts are on the other. We do this by multiplying both sides by dt and dividing both sides by A.	$\frac{1}{A} dA = r dt$
Integrate both sides. (If two quantities are equal, then their integrals must be equal.)	$\int \frac{1}{A} dA = \int r dt$
Anti-differentiate. (c is the difference between the two constants of integration.)	$\ln A = rt + c$
Exponentiate.	$A = e^{rt+c}$ $A = e^{c} e^{rt}$
Rename the constant e^c as P .	$A = Pe^{rt}$
When $t = 0$, $e^{rt} = 1$ so $A = P$, so P is the initial value.	

Other Differential Equations

Mathematicians need to solve many other types of differential equation – ones of the form $\frac{dy}{dx} = f(y)$, ones where the derivative is a function of both x and y, ones which include the derivative raised to powers, e.g. $\left(\frac{dy}{dx}\right)^2$ and ones which include higher-order derivatives like $\frac{d^2y}{dx^2}$. (You will meet $\frac{d^2y}{dx^2}$ in the next module.) If you major in mathematics or a related subject like physics or meteorology at university, you will spend some time learning techniques to solve such equations.

Solve

Q51 It was shown above that the solution to $\frac{dA}{dt} = rA$ is $A = Pe^{rt}$. This was done by separating variables. An alternative method is to take the reciprocal of both sides, then anti-differentiate with respect to A. Use this method to derive the solution.

Q52 Ice forms on a lake at a rate inversely proportional to the thickness of ice already there. The thickness of the ice is presently 10 mm and it is increasing at a rate of 3 mm/h. In how many hours time will it be 40 mm thick?

Q53 Solve $\frac{dy}{dx} = 2y + 12$, given that $y = 20$ when $x = 1$.

Q54 A projectile is fired vertically upwards. When $t = 0$, the height, h, is 10 m and its upwards velocity, v, is 50 m/s. While in flight it undergoes an acceleration from gravity and air resistance equal to $-9.8 - 0.1v$. Find

(a) the relation between v and t.

(b) the relation between h and t.

Q55 Solve the following differential equation by separating variables.

If $xy\frac{dy}{dx} - (1+x^2)(1-y^2) = 0$ and $y = 0$ when $x = 1$, find y when $x = 2$

Revise

Revision Set 1

Q61 A bacterial colony is growing at a rate in mg/h equal to 0.6 times the amount in mg already present. If there is 200 mg at 1 pm,

(a) How much will there be at 8 pm the same day

(b) When does the mass reach 1 g?

Q62 A snowball rolling down a hill picks up snow and gets bigger. Its rate of increase in volume is proportional to its volume. 5 seconds after starting to roll, its volume is 600 mL; 12 seconds later it has grown to 2.1 L. Find its volume another 7 seconds later still.

Q63 When a radioactive sample decays, the half-life is the time required for half of the initial sample to decay. Carbon-14 has a half-life of 5730 years.

(c) How much of a 2 g sample of Carbon-14 would remain after 9000 years?

(d) How long would it take for 20% of a Carbon-14 sample to decay?

Answers

Q1 6091 mg
Q2 (a) 1.61 g (b) 1:10 a.m. next day
Q3 $6356.24
Q4 79.7 g, 10:07 p.m. the next day
Q5 (a) 27° (b) 22.4 min
Q6 (a) 31° (b) 23 min
Q7 12.8°
Q8 445 mg
Q9 (a) 19.52 g (b) 19 030 years
Q10 (a) 141.8 L (b) 4.01 days (c) It will never completely empty
Q11 6.93 h

Q52 30 h
Q53 $y = 3.52e^{2x} - 6$
Q54 (a) $v = 148e^{-0.1t} - 98$ (b) $h = 1490 - 98t - 1480\,e^{-0.1t}$
Q55 $y = \pm 0.9938$

Q61 (a) 13 337 mg (b) 3:41 p.m.
Q62 4.36 L
Q63 (a) 0.67 g (b) 1844 years