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C6-11 Areas Under Curves

- area under curves
- area between curves
- trapezoidal rule
- Simpson's rule

Summary Learn Solve Revise Answers

Summary

The area between the graph of a function, $f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$ is equal to $\int_a^b f(x)\,dx$. For parts where $f(x)$ is negative, the algebraic area is negative, but the physical area is taken to be positive. The area between two curves is given by the area under the top one minus the area under the bottom one. Of course we have to calculate areas a bit at a time if $f(x)$ changes between positive and negative or if curves cross.

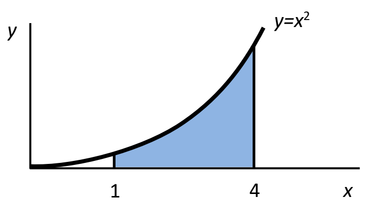
The trapezoidal rule and Simpson's rule are means of approximating the area under a curve and thus of approximating an integral which cannot be found by anti-differentiation.

Learn

Areas Under Curves

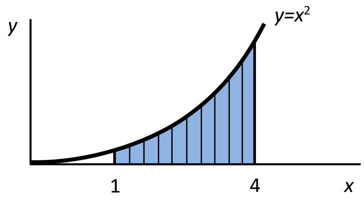
Integration can be used to find the area between the graph of a function and the x-axis or between two graphs. This is not a terribly useful thing to do, but it is a favourite of many teachers, text book authors and exam writers and in fact is often used as the main application of integration. So it's probably worth knowing about.

Let's suppose we wanted to know the area between the curve $y = x^2$, the x-axis, the line $x = 1$ and the line $x = 4$ (the blue area in the diagram to the right).



This area can be broken up into a series of vertical strips of width dx as shown in the next diagram to the right.

The area of each strip is the height multiplied by the width, i.e. $y\ dx$.



If we make the strips infinitely thin, then the total area is

$$\int_1^4 y\ dx = \int_1^4 x^2\ dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = 21$$

So the area is 21 square units. [A square unit is a square 1 unit in each direction on the graph.]

This should seem quite easy after the problems you've been solving in the last module.

The area between the x-axis and the graph of any function $y = f(x)$ between x-coordinates, a and b, is given by $\int_a^b f(x)\ dx$.

Practice

Q1 Find the area between the graph of $y = x^2$, the x-axis and the lines $x = -2$ and $x = 5$.

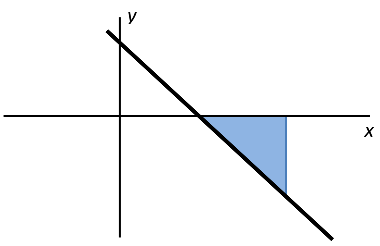
Q2 Find the area between the graph of $y = x^3 + 3x$, the x-axis and the lines $x = 0$ and $x = 4$.

Q3 Find the area between the graph of $y = {}^1/_x$, the x-axis and the lines $x = 2$ and $x = 8$.

Q4 Find the area between the graph of $y = 4 - x$, the x-axis and the lines $x = 4$ and $x = 6$.

Algebraic Area and Physical Area

You should have noticed that the last question gives a negative value for the area. This is because all the y-values are negative and the area is below the x-axis and therefore above the graph.



We talk about the algebraic area under a curve, which is what integration gives us. It will be positive if the graph is above the x-axis or negative if it's below the x-axis. We also talk about the physical area 'under' a curve, which is the geometric area of the blue shape. It is always positive. The physical area is the absolute value of the algebraic area.

In this last question, the algebraic area was –8; the physical area is 8.

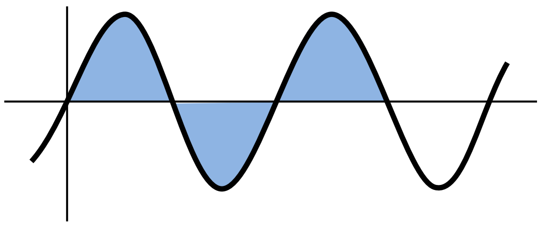
Now suppose we integrated to find the area between the x-axis and the line $y = 4 - x$ from $x = 0$ to $x = 8$. We would get

Area = $\int_0^8 4 - x\, dx = \left[4x - \frac{x^2}{2}\right]_0^8 = [4 \times 8 - \frac{8^2}{2}] - [4 \times 0 - \frac{0^2}{2}] = 0$

The algebraic area from $x = 0$ to $x = 4$ is 8, the algebraic area from $x = 4$ to $x = 8$ is –8. Adding these together makes 0.

So, if we need to find the physical area between a graph and the x-axis between $x = a$ and $x = b$, and the graph crosses the x-axis between a and b, we have to find out the x-coordinates where it crosses the x-axis (by solving $y = 0$), then calculate the physical area between the limits and each crossing separately, then add them together.

For example the graph of $y = \sin x$ is shown below.



To find the physical area between $x = 0$ and $x = 3\pi$ (the blue area), we have to find the areas from 0 to π, from π to 2π and from 2π to 3π separately, then add them.

Practice

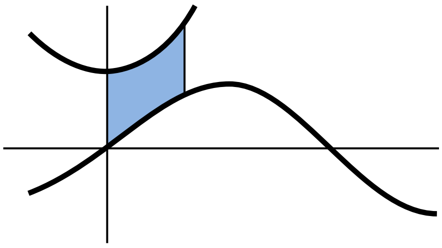
Q5 Find the algebraic area and the physical areas between the graph of $y = x^2 - 4$, the x-axis and the lines $x = 0$ and $x = 3$.

Q6 Find the algebraic area and the physical areas between the graph of $y = x^3 - 9x$, the x-axis and the lines $x = -5$ and $x = 4$.

Q7 Find the physical area between the graph of $y = e^x - 2$, the x-axis and the lines $x = 0$ and $x = 1$.

Areas Between Curves

Consider the graphs of $y = x^2 + 1$ and $y = \sin x$. They are shown plotted together below.

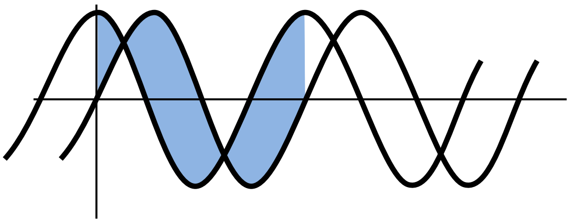


The physical area between the curves from $x = 0$ to $x = 1$ is shown in blue. It should be fairly easy to see that this is the area below the top curve minus the area below the bottom curve.

This will always be the case, even if one or both of the curves go below the x-axis. Try to work out why this is the case – think about subtracting negative areas.

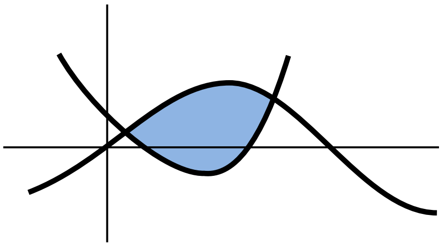
It won't apply, though, if the curves cross each other. In that case, there won't be a definite top curve and bottom curve. So, if they cross, we have to work out the areas between each crossing point (intersection), then add them all.

For example, the graph below is of $y = \sin x$ and $y = \cos x$.



The physical area between the curves from $x = 0$ to $x = 2\pi$ is shaded. To evaluate this, we have to find each of the three blue areas separately, then add them. To find the x-coordinates at the intersections, we equate the two functions: $\sin x = \cos x$ and solve.

Sometimes, two graphs will cross in two places giving a single finite area cut off between them, as in this diagram.



To find the cut-off area (the blue area in the diagram), we have to find the x-values of the two intersections by solving the equation produced by equating the two functions. Then we integrate both functions over that interval and find the difference. (Take the absolute value if negative.)

To find the algebraic area between two curves, you have to decide which is the top one, then just integrate them both over the whole interval and subtract the algebraic area under the bottom curve from the algebraic area under the top curve.

Practice

Q8 Find the physical area between the graph of $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \pi$.

Q9 Find the area cut off between the graphs of $y = x^2 - 2x$ and $y = 4 - x^2$.

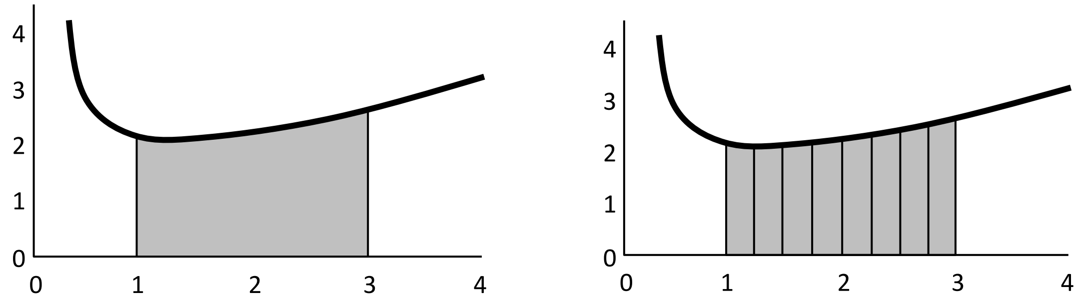
Q10 Find the algebraic area between the graphs of $y = 2 \sin x$ and $y = \cos x$, between $x = 0$ and $x = \pi$, taking $y = \cos x$ as the top graph.

Trapezoidal Rule

Not all functions can be anti-differentiated easily. Some can't be anti-differentiated at all. In such cases, it is possible to approximate the definite integral between given x-values by approximating the area under the curve. We can do this using the trapezoidal rule.

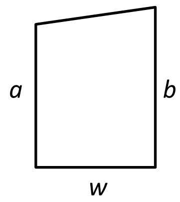
Suppose we wanted to find $\int_1^3 \frac{2^x}{x}\, dx$. This is hard to find algebraically, but it can be approximated using the trapezoidal rule.

As an area under the curve, the integral looks like the graph on the left below. The area can be divided into 8 strips as in the graph on the right.



We then assume that the top of each strip is a straight line, making each strip a trapezium. [The American for *trapezium* is *trapezoid*, hence the name of the rule.]

We then find the area of each trapezium, which is the sum of the left and right sides, divided by 2, then multiplied by the width of the strip. We then add the areas together.



$Area = \frac{a+b}{2} \times w$

As the outside strip sides are counted just once and all the others are counted twice, the area can be obtained by adding all the strip heights with all but the outside two doubled, then dividing by 2 and multiplying by the strip width.

The length of each vertical line can be obtained by finding the value of the function at that x-value. The calculation can be done with a table like this.

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
y	2	1.9027	1.8856	1.9220	2	2.1141	2.2627	2.4462	2.6667
Multiplier	1	2	2	2	2	2	2	2	1
Product	2	3.8054	3.7712	3.8440	4	4.2282	4.5254	4.8924	2.6667

Then we add the products to get 33.7333.

Then we divide by 2 and multiply by the width of the strips, i.e. 0.25.

$33.72 \div 2 \times 0.25 = 4.2167$

So $\int_1^3 \frac{2^x}{x}\,dx \approx 4.2167$

Practice

Q11 Use the trapezoidal rule to approximate the following definite integrals. Use 5 strips for each.

(a) $\int_2^4 x\,dx$ (b) $\int_2^3 t^2\,dt$ (c) $\int_0^5 x \sin x\,dx$

Q12 Find the integral of x^4 from $x = 1$ to $x = 2$ using calculus and using the trapezoidal rule. (Use just 4 strips.) Find the error in the trapezoidal rule result.

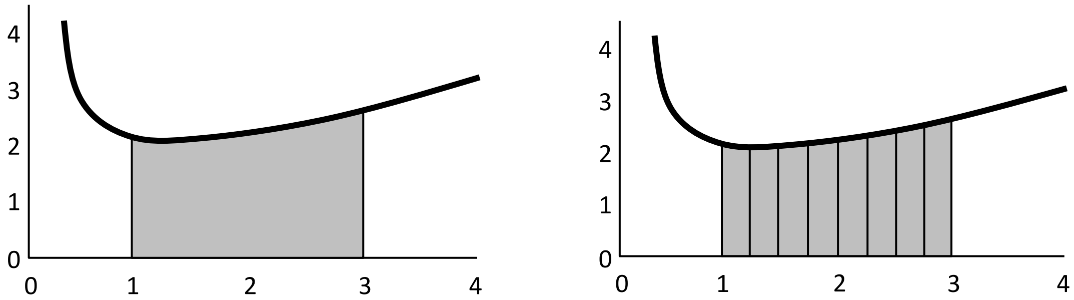
Simpson's Rule

Simpson's rule is another way of finding the area under a curve It is almost the same as the trapezoidal rule and just as easy to use.

The difference is that, instead of using 1 2 2 2 2 . . . . 2 2 2 1 as the multipliers, we use 1 4 2 4 2 . . . 4 2 4 1. Then instead of dividing by 2, we divide by 3.

Because the second and second-last multipliers must be 4s, we must use an odd number of lines (i.e. an even number of slices).

Let's apply Simpson's rule to the same problem, i.e. to find $\int_1^3 \frac{2^x}{x}\,dx$.



x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
y	2	1.9027	1.8856	1.9220	2	2.1141	2.2627	2.4462	2.6667
Multiplier	1	4	2	4	2	4	2	4	1
Product	2	7.6108	3.7712	7.6880	4	8.4564	4.5254	9.7848	2.6667

Then we add the products to get 50.5036.

Then we divide by 3 and multiply by the width of the strips, i.e. 0.25.

$50.49 \div 3 \times 0.25 = 4.2086$

So $\int_1^3 \frac{2^x}{x}\,dx \approx 4.2086$

Simpson's rule is generally a bit more accurate than the trapezoidal rule. The difference is that Simpson's rule calculates the area of each strip assuming the curve to be a quadratic whose parameters are affected by the points in the far corners of the adjacent strips.

The mathematics behind it is tricky and you probably don't need to worry about it. But using the rule is simple and very much like using the trapezoidal rule.

The trapezoidal rule will give an exact result for the integral of a linear function. This should be obvious if you think about it. Simpson's rule will give an exact result for the integral of any linear, quadratic or cubic function.

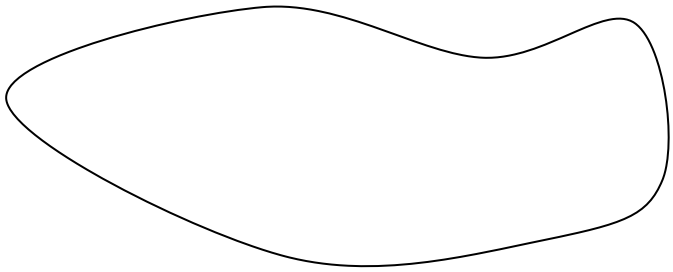
Practice

Q13 Use Simpson's rule to find the following definite integrals. Use 8 strips for (a) and (b), 6 strips for (c).

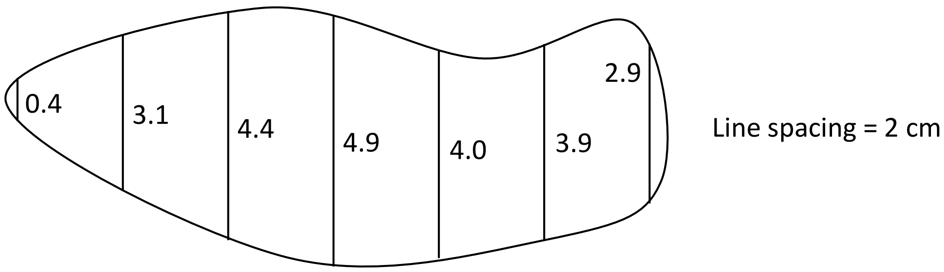
(a) $\int_0^4 x \sin x \, dh$ (b) $\int_2^3 p^2 \times 2^p \, dp$ (c) $\int_2^5 x^x \, dx$

Areas of Blobs

The trapezoidal rule and Simpson's rule are mostly used to find areas under curves and definite integrals where the function cannot be easily integrated. However, they can also be used to find the areas of irregular shapes like the one below.



First, we draw evenly spaced parallel lines across it. The outside lines should be as close as possible to the ends of the shape and, if using Simpson's rule, there should be an odd number of lines. We measure the length of each line and the spacing between them.



Then we use the rule in the same way as before, using the length of each line.

With the trapezoidal rule:

Length (cm)	0.4	3.1	4.4	4.9	4.0	3.9	2.9
Multiplier	1	2	2	2	2	2	1
Product	0.4	6.2	8.8	9.8	8.0	7.8	2.9

Total: 43.9 Line spacing = 2 cm. Area $43.9 \div 2 \times 2 = 43.9$ cm^2.

With Simpson's rule:

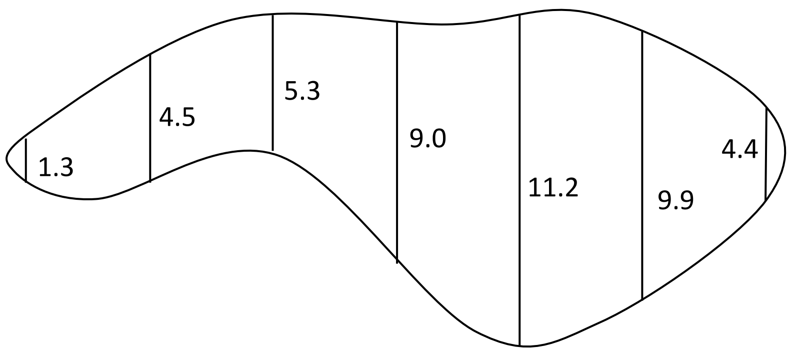
Length (cm)	0.4	3.1	4.4	4.9	4.0	3.9	2.9
Multiplier	1	4	2	4	2	4	1
Product	0.4	12.4	8.8	19.6	8.0	15.6	2.9

Total: 67.7 Line spacing = 2 cm. Area 67.7 ÷ 3 × 2 = 45.1 cm².

If the end lines aren't close to the ends of the shape, it may be necessary to add a little bit to allow for the extra bits of the shape.

Practice

Q14 Use the trapezoidal rule and Simpson's rule to find the area of the shape in P1, reproduced below (measurements in metres, line spacing = 5 m).



Solve

Q51 Find the integral of x^4 from $x = 1$ to $x = 2$ using calculus, using the trapezoidal rule and using Simpson's rule. (Use just 4 strips in each case.) Compare the error in the Simpson's rule result to the error in the trapezoidal rule result.

Revise

Revision Set 1

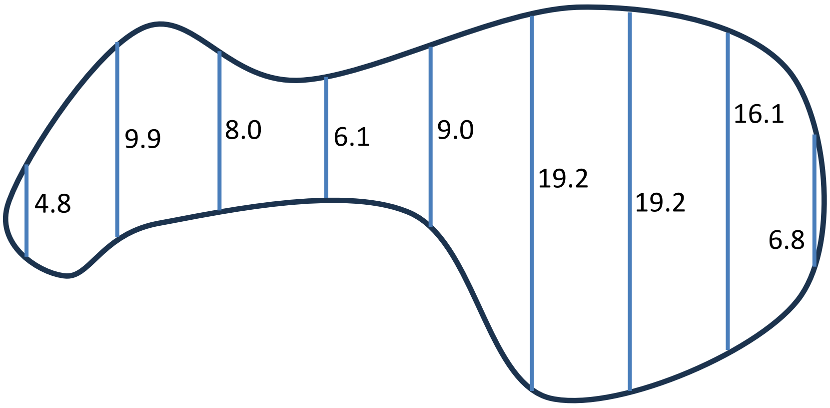
Q61 Find the physical area between the graph of $y = {}^1/_x - 1$ and the x-axis between $x = ½$ and $x = 2$.

Q62 Find the physical area cut off between $y = x + 2$ and $y = x^2$.

Q63 Use the trapezoidal rule to approximate $\int_0^4 x2^x dx$ using 8 strips.

Q64 Use Simpson's rule to find $\int_0^4 x2^x dx$ using 8 strips.

Q65 Use the trapezoidal rule to find the area of the shape below. The line lengths are in metres and the strip width is 5 m.



Answers

Q1 $44\frac{1}{3}$ Q2 88 Q3 1.39 Q4 –2

Q5 Algebraic –3, physical $7\frac{2}{3}$ Q6 Algebraic –51¾, physical 159¾ Q7 0.491

Q8 $2\sqrt{2}$ Q9 9 Q10 –4

Q11 (a) 6 (b) 6.34 (c) –2.34

Q12 Calculus: 6.1667 Trapezoidal: 6.3457 (error: 0.179)

Q13 (a) 1.86 (b) 306 (c) 1256

Q14 trapezoidal rule: 466.5 m^2 Simpson's rule : 482 m^2

Q51 Calculus: 6.1667 Trapezoidal: 6.3457 (error: 0.179) Simpsons: 6.201 (error: 0.034)

Q61 –2.88 Q62 4½ Q63 62.3 Q64 61.1 Q65 372 m^2