M1 Maths

C6-1 Velocity Graphically

- displacement vs time graphs: high/low, rising/falling, steeper/flatter
- gradient = velocity
- gradient of secant = average velocity
- gradient of tangent = instantaneous velocity

Summary Learn Solve Revise Answers

Summary

On a line graph of distance vs time,

- the higher the graph, the greater the displacement;
- a rising graph indicates increasing displacement, a falling graph decreasing displacement;
- the steeper the graph the faster the change in displacement.

The gradient of the graph is equal to the velocity. A negative gradient means a negative velocity, i.e. travel in the opposite direction.

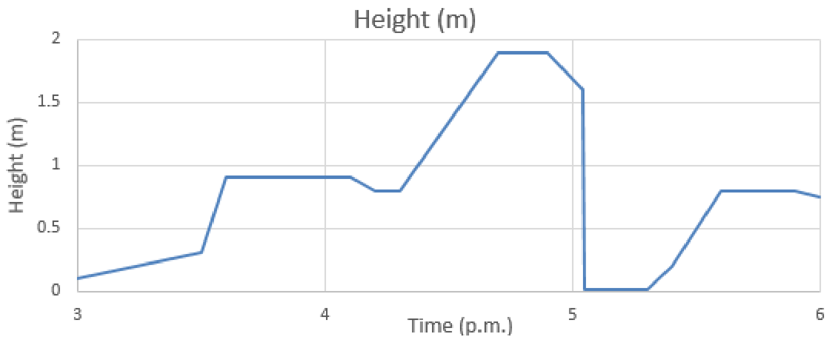
The gradient of a secant to the graph (a line joining two points on the graph) is equal to the average velocity over that time interval.

The gradient of the tangent to the graph at a point is equal to the instantaneous velocity at that time.

Learn

Distance vs time graphs: high/low, rising/falling, steeper/flatter

The graph below shows the height of a snail on a drain pipe between 3 pm and 6 pm one afternoon.



Practice

Q1 How high was the snail at

(a) 3 pm (b) 4 pm (c) 5 pm

Q2 Roughly when was the snail

(a) highest (b) lowest

Q3 Was the snail going up, down or neither at

(a) 3:10 pm (b) 4 pm (c) 5 pm (d) 6 pm

Q4 (a) How many times did the snail stop moving?

(b) When was the snail going up fastest?

(c) When was it going down fastest?

(d) What do you think happened just after 5 pm?

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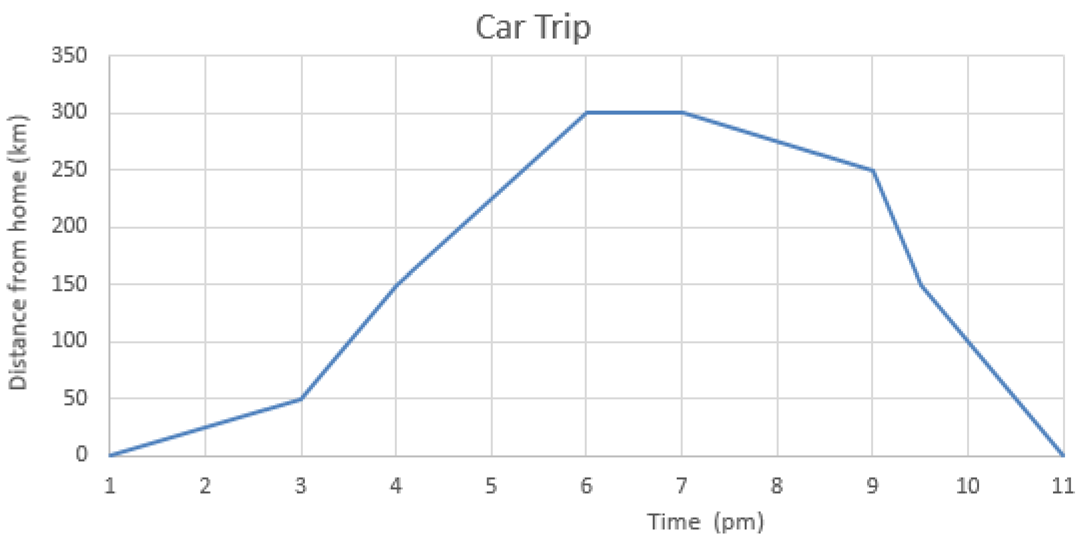
After doing Q1-4, you should realise that:

- the higher the graph, the higher the snail
- when the graph is going up the snail is going up and vice versa
- the steeper the graph, the faster the snail is going up or down.

That probably isn't anything new to you.

Gradient = Velocity

The following is a graph of distance from home vs. time for a car trip.



Practice

Q5 (a) How far did they travel between 1 pm and 3 pm?

(b) How fast were they travelling between 1 pm and 3 pm?

(c) What is the gradient of the section of the graph between 1 pm and 3 pm?

You should have got an answer of 25 for part (c). The gradient of the graph doesn't look like 25, but this is because the scales on the two axes are very different. Of course, we shouldn't go by looks, though, but by the numbers. Over that interval, the rise was 50 (km) and the run was 2 (hours), so the gradient, being $^{rise}/_{run}$, is $^{50}/_{2}$ or 25.

Q6 (a) How fast were they travelling between 3 pm and 4 pm?

(b) What is the gradient of the section of the graph between 3 pm and 4 pm?

(c) How fast were they travelling between 4 pm and 6 pm?

(d) What is the gradient of the section of the graph between 4 pm and 6 pm?

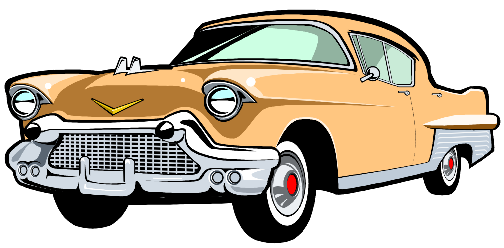
(e) How fast were they travelling between 6 pm and 7 pm?

(f) What is the gradient of the section of the graph between 6 pm and 7 pm?

(g) What is the relationship between their speed and the gradient of the graph?

Q7 (a) How fast were they travelling between 7 pm and 9 pm?

(b) What is the gradient of the graph between 7 pm and 9 pm?



Velocity – a Vector

For Q7(a), you probably got 25 km/h and for Q7(b), you should have got –25. Unlike in Q6, the speed and the gradient aren't the same: one is positive and one is negative.

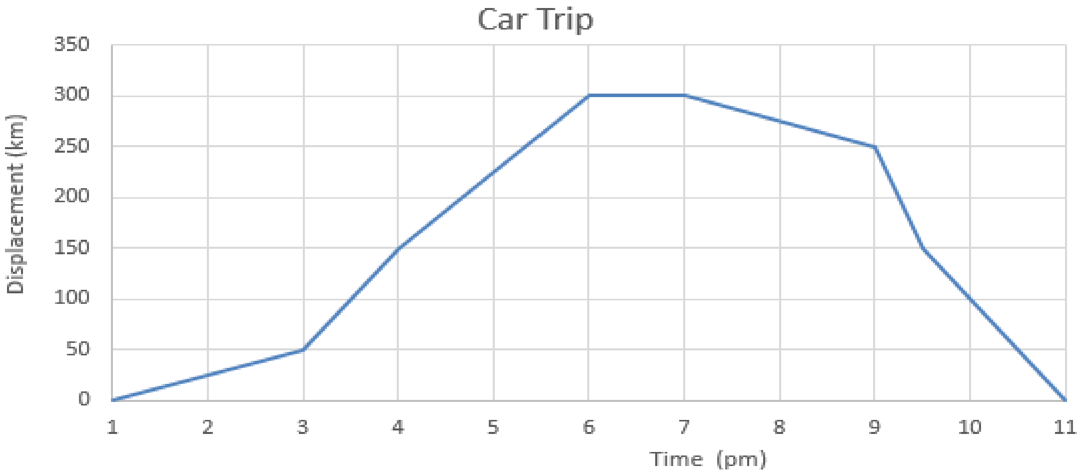
We can get around this by talking about velocity rather than speed. Velocity is a vector and so has a magnitude and a direction. The velocity between 7 pm and 9 pm has magnitude 25 km/h and direction 'back towards home'. When dealing with vectors in just one dimension, like we are here, we can define one direction as positive and the opposite direction as negative. It's normal to call 'up the graph' (i.e. increasing distance from home) positive and 'down the graph' (decreasing distance from home) negative. If we do that, then the velocity between 7 pm and 9 pm is –25 km/h. This means that velocity and gradient are then the same, whichever direction the car is heading. From here on, we will talk about velocity rather than speed.

As mentioned, velocity is a vector (it has magnitude and direction). By comparison, speed is a scalar (it has just magnitude). So a speed might be 50 km/h; the velocity might be 50 km/h towards home or –50 km/h. The speedometer on a car measures speed – it takes no notice of the direction. An instrument that told you that you were going 50 km/h away from home would be a velocityometer.

To be strictly consistent, we should also talk about displacement rather than distance from home. Displacement is a vector (whereas distance is a scalar). Displacement could be negative if we went the other direction from home, but you can't be a negative distance from home. Velocity is the rate of change of displacement, i.e. the change in displacement in a unit of time. Between 7 pm and 9 pm, the displacement changed from 300 km to 250 km, a change of –50 km in 2 hours, giving a velocity of –25 km/h.

Practice

Here are some more questions on the same car trip graph. The graph is reproduced here for convenience.



Q8 (a) What was the velocity between 7 pm and 9pm?

(b) What is the gradient of the graph between 7 pm and 9 pm?

(c) What was the velocity between 9 pm and 9:30 pm?

(d) What is the gradient of the graph between 9 pm and 9:30 pm?

(e) What was the velocity between 9:30 pm and 11 pm?

(f) What is the gradient of the graph between 9:30 pm and 11 pm?

Q9 (a) When was the speed greatest?

(b) When was the speed least? [Remember speed cannot be negative.]

(c) When was the velocity greatest?

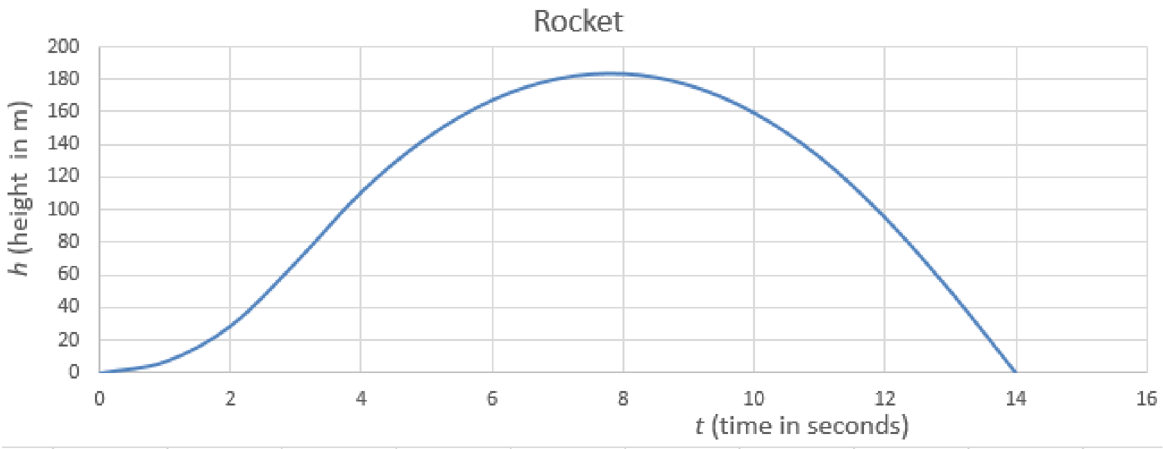
(d) When was the velocity least? [Remember large negative numbers are lower than small positive numbers.]

After doing Q5-8, you should be aware that the velocity at any time is given by the gradient of the graph at that time. This is because the rise in the graph is the change in displacement and the run is the change in time.

$$velocity = \frac{change\ in\ displacement}{change\ in\ time} = \frac{rise}{run} = gradient$$

Gradient of Secant = Average Velocity

Below is a graph of height vs. time for a firework rocket.



This is different from previous graphs in that the line is curved because the velocity is changing constantly.

Practice

Q10 What was the height of the rocket at

(a) $t = 0$ (b) $t = 2$ (c) $t = 6$ (d) $t = 12$

Q11 What was the average velocity of the rocket between $t = 2$ and $t = 6$

Hopefully you realised that the average velocity between $t = 2$ and $t = 6$ is the change in h over that interval divided by the change in t over that interval.

In calculus, it is common to use Δh for the change in h and Δt for the change in t. Δ, pronounced *delta*, is the Greek letter d and stands for difference – difference in h or difference in t.

The height at $t = 2$, $h(2) = 29$. The height at $t = 6$, $h(6) = 167$.

$\Delta h = 167 - 29 = 138$

$\Delta t = 6 - 2 = 4$

$\frac{\Delta h}{\Delta t} = \frac{138}{4} = 34.5$

So the average velocity was about 35 m/s

Practice

Q12 What was the average velocity of the rocket between

(a) $t = 2$ and $t = 8$

(b) $t = 2$ and $t = 5$

(c) $t = 0$ and $t = 2$

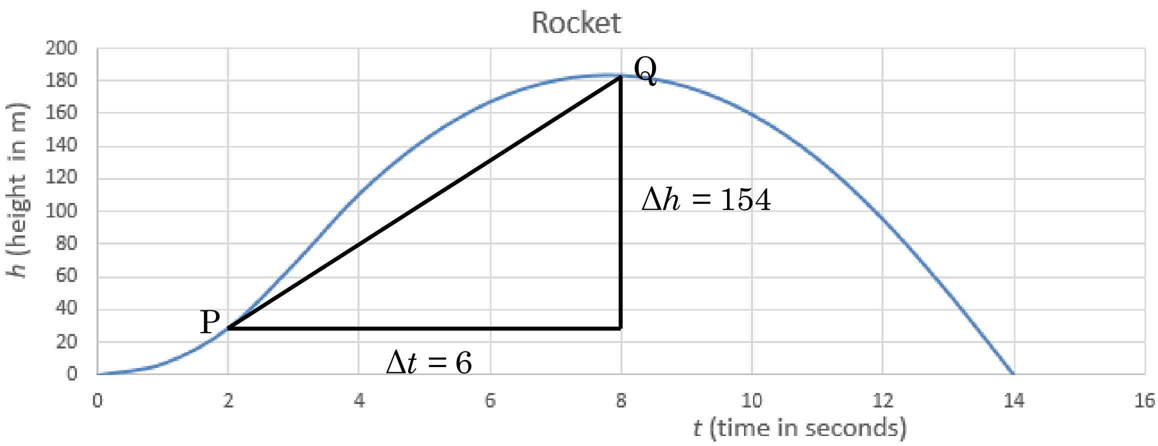
(d) $t = 4$ and $t = 10$

(e) $t = 6$ and $t = 12$

(f) $t = 0$ and $t = 14$

Note that the average velocity between $t = 6$ and $t = 12$ is negative because Δh was negative over that period – the displacement decreased rather than increasing. The average velocity between $t = 0$ and $t = 14$ was zero, because $\Delta h = 0$ over that interval.

When we find the average velocity between $t = 2$ and $t = 8$, we have to find Δh and Δt. We could draw these on the graph like this:



We can then see that $\frac{\Delta h}{\Delta t}$ is $\frac{rise}{run}$ for the triangle and therefore the gradient of the line PQ.

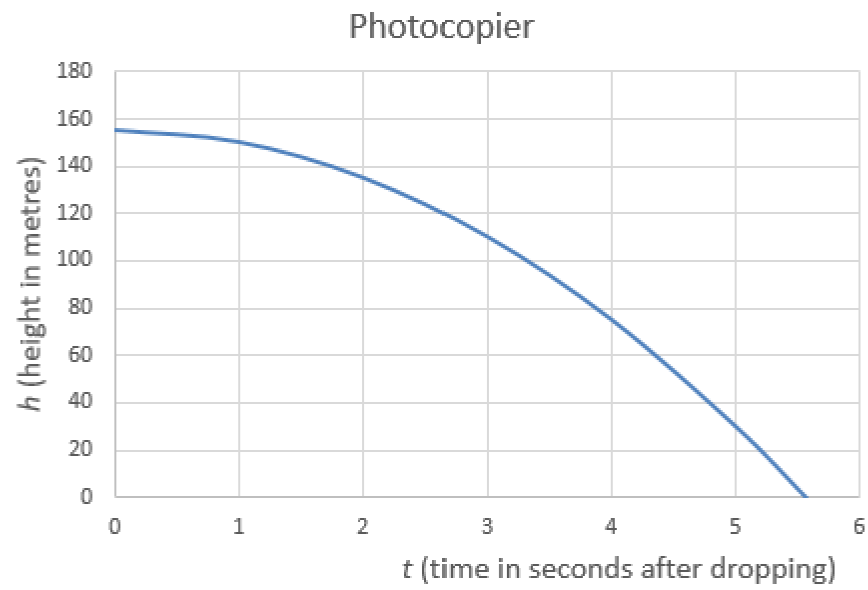
PQ is called a secant to the curve. A secant is a line joining two points on the curve, in this case P and Q. The word *secant* is Latin for *cutting*, because, if the line is continued it cuts across the curve in two places.

This leads us to an important result:

The gradient of the secant between two points on a displacement-time graph is equal to the average velocity between those two times.

Practice

Q13 A photocopier is dropped out of the window from the 50th floor of a tall building. The relation between height above the ground and time is given in the graph below.



For each of the following intervals:

(i) draw the secant (or imagine it)

(ii) find h at the beginning and end of the interval

(iii) find Δh and Δt

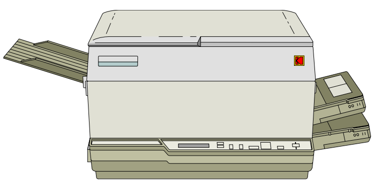
(iv) find the average velocity (note that, as the height is decreasing, the velocities will all be negative)

(a) $t = 1$ to $t = 3$ (b) $t = 0$ to $t = 4$ (c) $t = 0$ to $t = 2$

(d) $t = 2$ to $t = 3$ (e) $t = 2$ to $t = 4$ (f) $t = 2$ to $t = 5$

Q14 (a) When was the photocopier moving fastest?

(b) Did the photocopier go straight down or was it thrown sideways? Explain.



Gradient of Tangent = Instantaneous Velocity

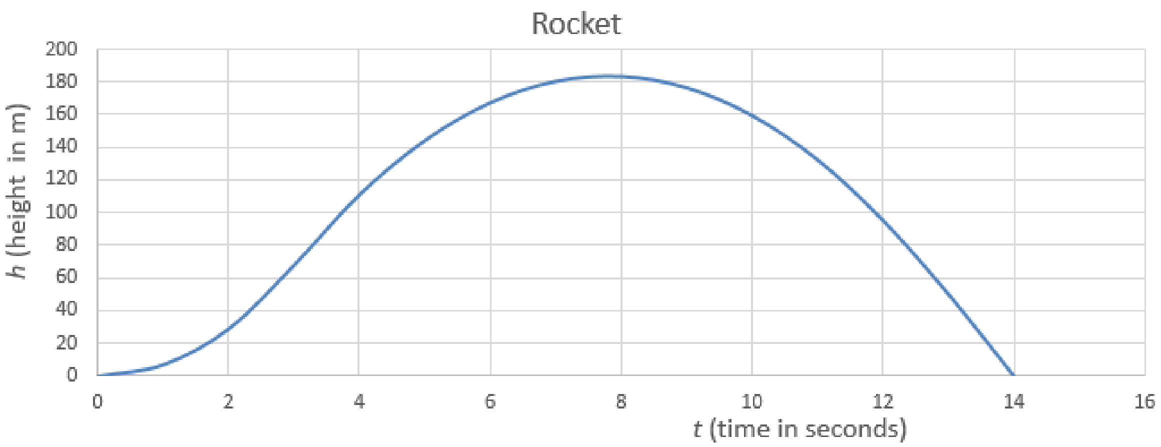
We have found velocities when the velocity is constant (straight-line sections on the graph). The velocity is the gradient of that straight-line section.

We have also found average velocities over periods of time when the velocity varies (curved section on the graph). The velocity is the gradient of the secant.

Now we will look at finding the instantaneous velocity at a point in time when the velocity is varying.

When you drive a car around the suburbs or on a race track, the velocity varies continuously. Yet your speedometer will show you the speed (magnitude of the velocity) at any time. What it shows is not the average speed over a period of time, but the speed at a point in time.

Let's have another look at that rocket.

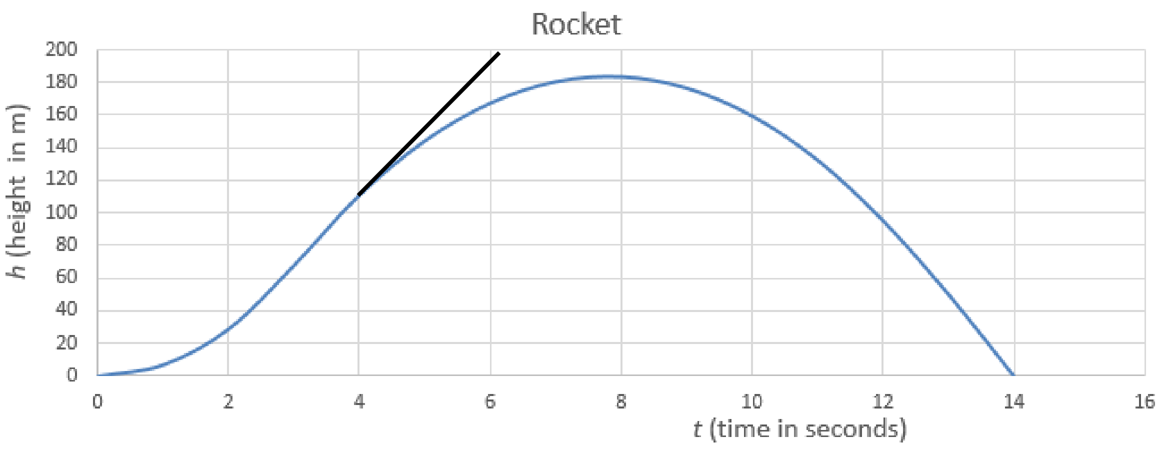


The rocket has a velocity at any given point in time, though it changes as time changes. Likewise the graph has a gradient at any given value of t, though it changes as t changes.

Suppose we wanted to know the velocity of the rocket when $t = 4$. We can see that it's going quite fast upwards, but how can we get a value? We saw how to get the average velocity over a period, but here, there is no period to find the average over. We are dealing with a single point in time.

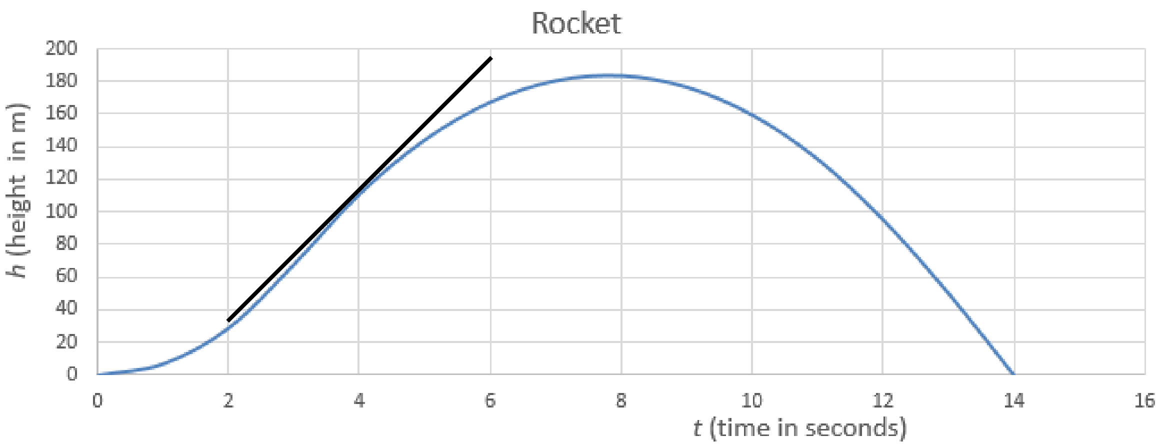
Before reading on, see if you can think of a way of doing it.

This is what we do. We imagine that, from $t = 4$ onwards, the velocity stays the same and we re-draw that part of the graph accordingly. We realise that the new graph would continue from $t = 4$ with the same gradient, i.e. at the same angle or in the same direction. The result should look like this.



We now have a straight line with a gradient we can measure and this gradient will be the same as the instantaneous velocity at $t = 4$.

Actually, the easiest and most accurate way of drawing this line is to continue it in both directions from $t = 4$. Do this by turning the paper round so the curve is horizontal at that point and curving upwards to the left and right. Then place a ruler under the curve so that the curve runs along it at $t = 4$. Then draw the line along the edge of the ruler in both directions. It should come out something like this.



This longer line also makes it easier to measure the gradient.

To find the gradient, we just pick two points on the line, find Δh and Δt and divide them. If we pick (2, 32) and (6, 192) (the ends of the black line), then $\Delta h = 160$ and $\Delta t = 4$. Therefore $\frac{\Delta h}{\Delta t} = \frac{160}{4} = 40$.

So the instantaneous velocity at $t = 4$ is about 40 m/s.

Note that this figure won't be exact because drawing the straight line is subject to error.

The straight line is called a *tangent*. Tangent is Latin for *touching*. It is called this because it touches the curve without crossing it.

So this technique involves drawing a tangent to the curve at the t value we need, then finding the gradient of the tangent.

This is our last important result:

> The gradient of a tangent at a point on a displacement-time graph is equal to the instantaneous velocity at that point.

Practice

Q15 Use the tangent technique and the rocket graph above to find the instantaneous velocity of the rocket when

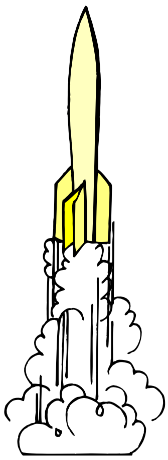
(a) $t = 2$ (b) $t = 6$ (c) $t = 9$ (d) $t = 11.5$

Solve

Q51 The height (in metres), h, of a rocket at time (in seconds), t, is given by the formula $h = 4t^2$. By drawing a graph of height vs time and finding an appropriate gradient, find the velocity when $t = 5$.

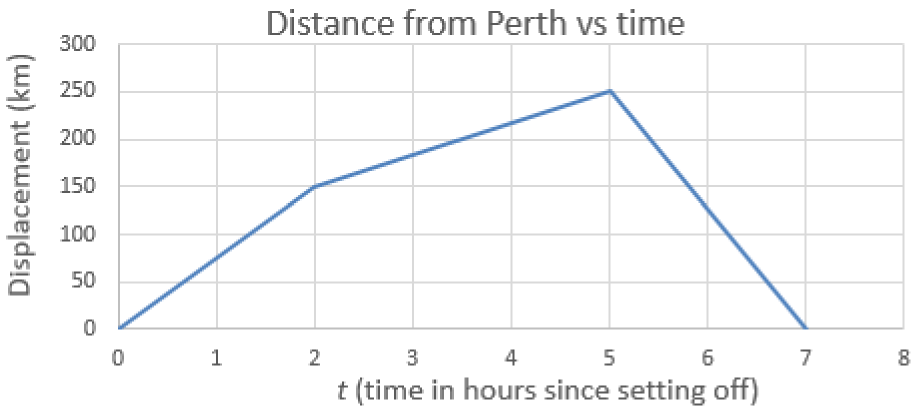
Q52 At what time would the rocket in Q51 have a velocity of 25 m/s?

Q53 The height (in metres), h, of a rocket at time (in seconds), t, is given by the formula $h = 2.718^t$. Find the height and the velocity at a few different times, then find a formula for the velocity in terms of time.

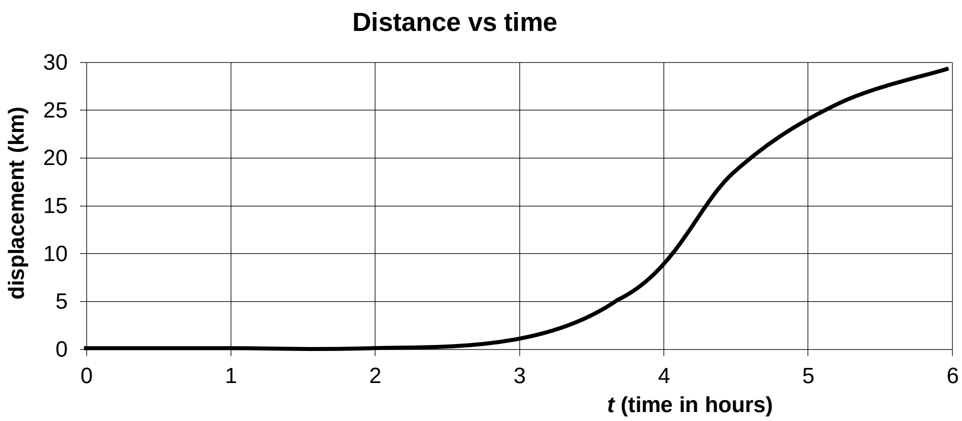


Revision Set 1

Q61 Using the following graph, find the velocity at $t = 1$.



Q62 Using the secant method on the following graph of displacement vs time (t), find the average velocity between $t = 4$ and $t = 6$.

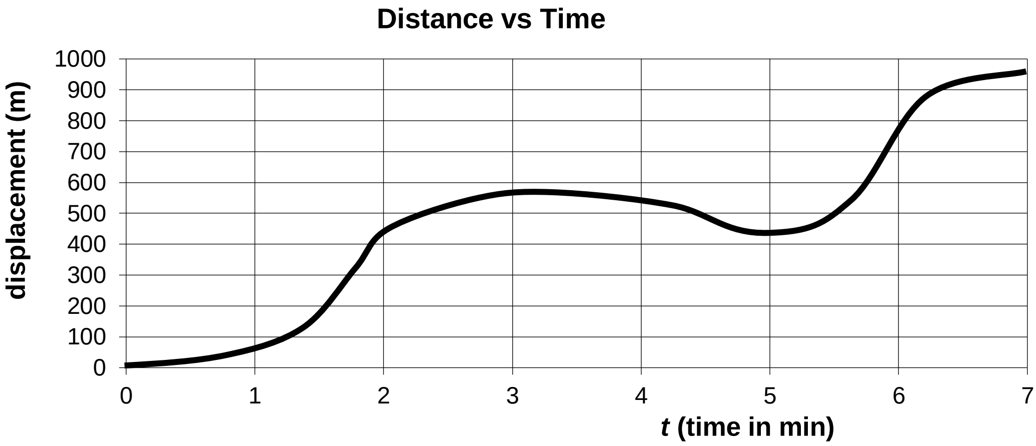


Q63 Use the tangent method on the displacement vs time graph above to find the instantaneous velocity at $t = 4$.

Revision Set 2

Q71 Using the graph in Q61, find the velocity between $t = 2$ and $t = 4$.

Q72 The following graph shows the relation between distance from the starting line and time for a slightly inebriated athlete. Use the secant method to find the average velocity (in m/min) over the first 2 minutes (from $t = 0$ to $t = 2$).



Q73 Use the graph above and the tangent method to find the instantaneous velocity (in m/min) at $t = 4$.

Revision Set 3

Q81 Using the graph in Q61, find the velocity at $t = 6$.

Q82 Using the graph in Q72 and the secant method, find the average velocity (in m/min) between $t = 1$ and $t = 5$.

Q83 Using the graph in Q72 and the tangent method, find the instantaneous velocity (in m/min) at $t = 6$.

Answers

Note that your answers to some of the questions may be slightly different from those given.

Q1	(a) 0.1 m	(b) 0.9 m	(c) 1.7 m	
Q2	(a) 4:40 to 4:50	(b) 5:03 to 5:15		
Q3	(a) up	(b) neither	(c) down	(d) down
Q4	(a) 4 times	(b) 3:30 to 3:35	(c) 5:03	(d) it fell off
Q5	(a) 50 km	(b) 25 km/h	(c) 25	
Q6	(a) 100 km/h	(b) 100	(c) 75 km/h	(d) 75
	(e) 0 km/h	(f) 0	(g) speed = gradient	
Q7	(a) 25 km/h	(b) –25		
Q8	(a) –25 km/h	(b) –25	(c) –200 km/h	(d) –200
	(e) –100 km/h	(f) –100		
Q9	(a) 9 to 9:30 pm	(b) 6 – 7 pm	(c) 3 – 4 pm	(d) 9 – 9:30 pm
Q10	(a) 0 m	(b) 29 m	(c) 167 m	(d) 93 m
Q11	35 m/s			
Q12	(a) 26 m/s	(b) 38 m/s	(c) 15 m/s	(d) 8 m/s
	(e) –36 m/s	(f) 0 m/s		
Q13	(a) (ii) 150, 110 m	(iii) –40, 2 m	(iii) –20 m/s	
	(b) (ii) 155, 75 m	(iii) –80, 4 m	(iii) –20 m/s	
	(c) (ii) 155, 135 m	(iii) –20, 2 m	(iii) –10 m/s	
	(d) (ii) 135, 110 m	(iii) –25, 1 m	(iii) –25 m/s	
	(e) (ii) 135, 75 m	(iii) –60, 2 m	(iii) –30 m/s	
	(f) (ii) 135, 30 m	(iii) –105, 3 m	(iii) –35 m/s	

Q14 (a) when it hit the ground
(b) It went straight down: the question says it was dropped. The horizontal axis of the graph is time, not horizontal displacement.

Q15 (a) 30 m/s (b) 18 m/s (c) –12 m/s (d) –37 m/s

Q51 40 m/s
Q52 $t = 3.1$
Q53 $velocity = 2.718^t$

Q61 75 km/h
Q62 10 km/h
Q63 15 km/h

Q71 33 km/h
Q72 215 m/min
Q73 –60 m/min

Q81 –125 km/h
Q82 93 m/min
Q83 670 m/min