M1 Maths

A6-6 Loci and Conic Sections

- loci
- conic sections

Summary Learn Solve Revise Answers

Summary

A locus is the set of all points satisfying a given condition.

Conic sections are curves formed by the intersection of a cone and a plane. Depending on the angle between the cone's axis and the plane, the section may be a circle (eccentricity = 0), an ellipse ($0 < e < 1$), a parabola ($e = 1$), or a hyperbola ($e > 1$).

Conic sections occur in many situations not obviously related to cones.

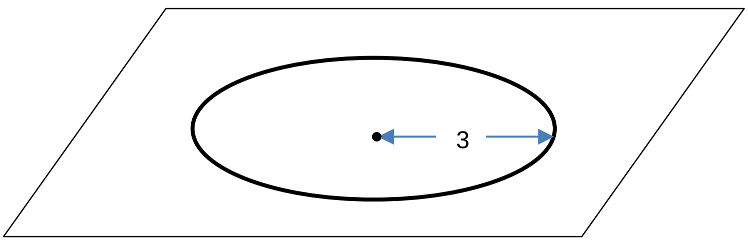
They can be expressed algebraically using coordinate geometry.

Learn

Loci

Loci is the plural of *Locus*. A locus is the set of all points satisfying a given condition.

For example, the set of all points in a plane which are 3 units from a given point on that plane is a circle of radius 3 around that given point.



The locus of all points equidistant from two points in 3-d space is the plane half way between those points and perpendicular to the line joining them.

Practice

Q1 Describe the following loci:

(a) All points in a plane equidistant from two points in the plane

(b) All points ≤ 3 cm from a given point in 3-d space

(c) All points closer to Point A than to point B on a plane

(d) All points on a bearing of 060° from the origin on the Cartesian plane

(e) All points on the Cartesian plane with y-coordinate equal to twice the x-coordinate

(f) All points on the Cartesian plane twice as far from the y-axis as from the point (0, 2)

(g) All points on the Cartesian plane equidistant from the x-axis and the point (0, 2)

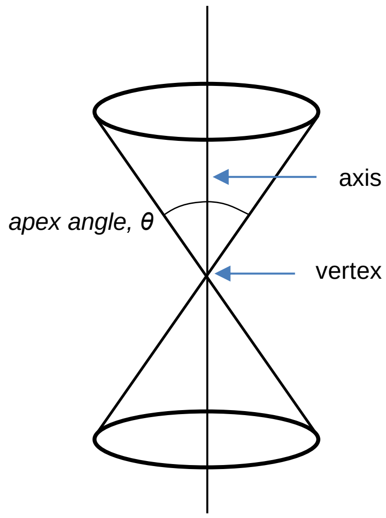
(h) All points in 3-d space twice as far from point A as from Point B

(i) All points on the Cartesian plane for which $|x| + |y| = 4$

(j) All points on the Cartesian plane for which $x + 1 = 2x + 1$

Cones

You would be familiar with a cone: it's the shape of a witch's hat. For this unit, we will define a cone more rigorously (and slightly differently).



A cone is the set of all lines passing through a given point (the vertex) at a given angle, $\theta/2$, to a given line (the axis).

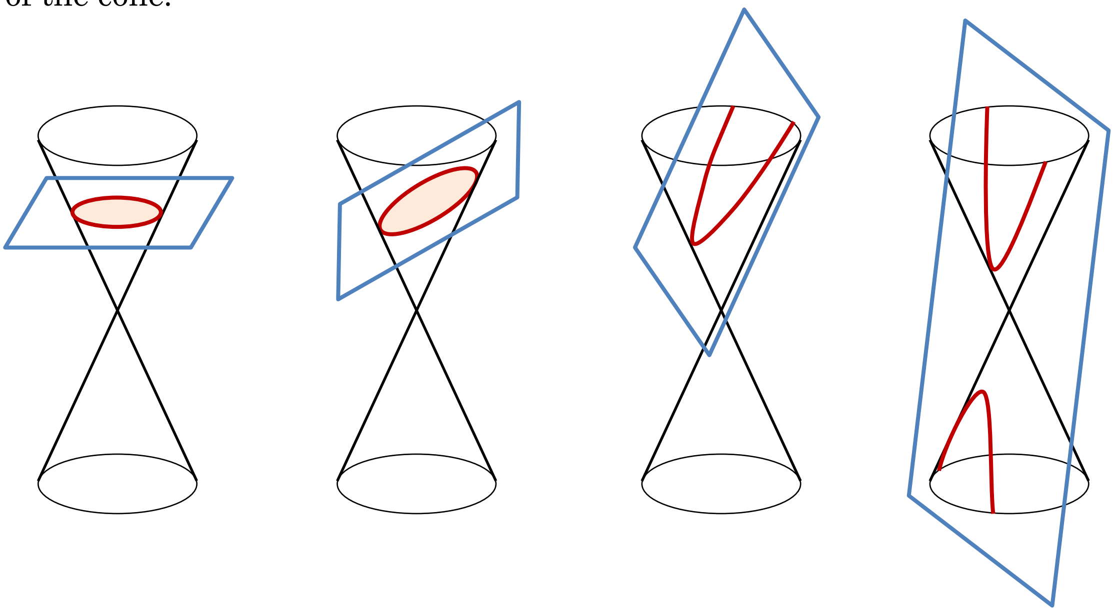
If $\theta/2$ is 30°, then the apex angle will be 60°.

Now, if you think about it, this cone is basically the same shape as the witch's hat, but differs from it in a couple of ways. Firstly, it has infinite height and no base; secondly, it has two parts (called *nappes*) joined at their vertices.

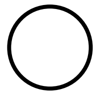
Conic Sections

If a plane intersects a cone, the shape of the trace of the cone on that plane is called a conic section. The word *section* comes from the word *to cut* and is used because the plane cuts through the cone.

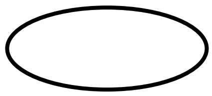
Different shaped curves can result, depending on the angle between the plane and the axis of the cone.



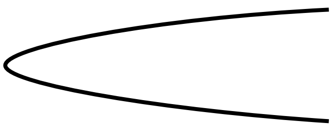
When the plane is perpendicular to the axis, we get a circle, which is round.



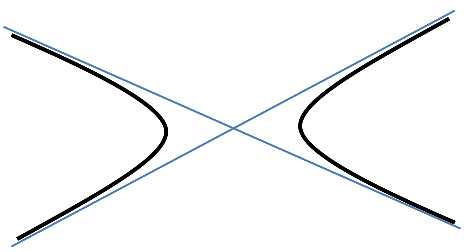
As the plane tilts, the circle is stretched in one direction relative to the other to make an ellipse.



When the plane reaches the angle θ from the axis, the ellipse becomes infinitely long in one direction. Then it is a parabola. An infinite distance from the vertex, the arms of a parabola are parallel.



Continuing to tilt the plane closer to the axis, the shape becomes a hyperbola. In a hyperbola, the arms approach a certain angle from one another, but never get close to being parallel. A hyperbola can be pictured as a curve that gets closer and closer to a cross (pair of asymptotes) as we get further and further from the vertex.



Natural Occurrences of Conic Sections

Conic sections occur frequently in nature.

Torch: If you shine straight at a wall, it will produce a circular lighted patch. Change the angle of the torch and the circle will become an ellipse, then a parabola, then a hyperbola. [Note that you would only see both sections of the hyperbola if the torch shone from both ends.]

Orbits: If a small body is in orbit around a larger body, the trajectory of its orbit will be a conic section. The orbit of the Earth around the Sun is almost a circle, though it is slightly elliptical, being slightly closer the Sun in January than July. Pluto has a more elliptical orbit, being closer to the Sun than Neptune for part of its orbit. Hayley's Comet has a very elliptical (almost parabolic) orbit. If a comet passes through the solar system and is deflected by the Sun's gravity, then keeps going never to return, then its orbit is hyperbolic.

It was Kepler's discovery that the planets orbited the sun in elliptical orbits that led Newton to develop the Law of Gravitation and calculus in order to explain this observation.

Shadow tracks: If you suspend a ball in the air and plot the position of its shadow on the ground over a day, the path will be a conic section. It will be a circle at the North or South Pole, an ellipse between the pole and the highest latitude that experiences the midnight sun on that date, a parabola at that latitude and a hyperbola elsewhere. This can be explained in terms of a plane (the ground) cutting a cone. You might like to think further about this.

Reciprocal functions have graphs which are hyperbolas. As the asymptotes (the x and y axes) are perpendicular, this type of hyperbola is called a rectangular hyperbola.

Locus Definition of Conic Sections and Eccentricity

We have defined conic sections in terms of cuts through a cone. But a conic section can also be defined as the locus of a point whose distance from a fixed point (the focus) is equal to its perpendicular distance from a fixed line (the directrix) multiplied by a constant.

The constant is called the eccentricity, e, of the conic section.

If $e = 0$, we have a circle

If $0 < e < 1$, we have an ellipse

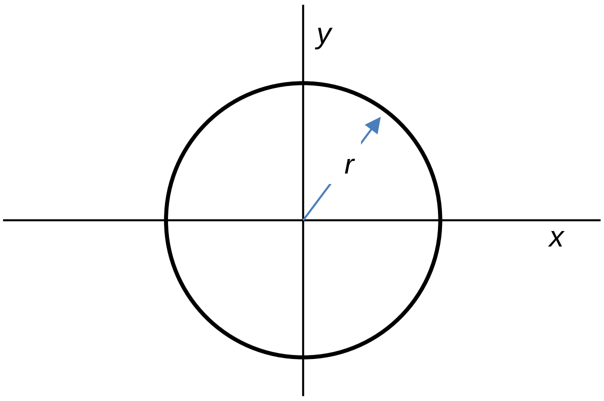
If $e = 1$, we have a parabola

If $e > 1$, we have a hyperbola

Because e is fixed for circles and parabolas, all circles have the same shape and all parabolas have the same shape; because e can vary over a range for ellipses and hyperbolas, ellipses and hyperbolas vary in shape.

Coordinate Geometry of Conic Sections

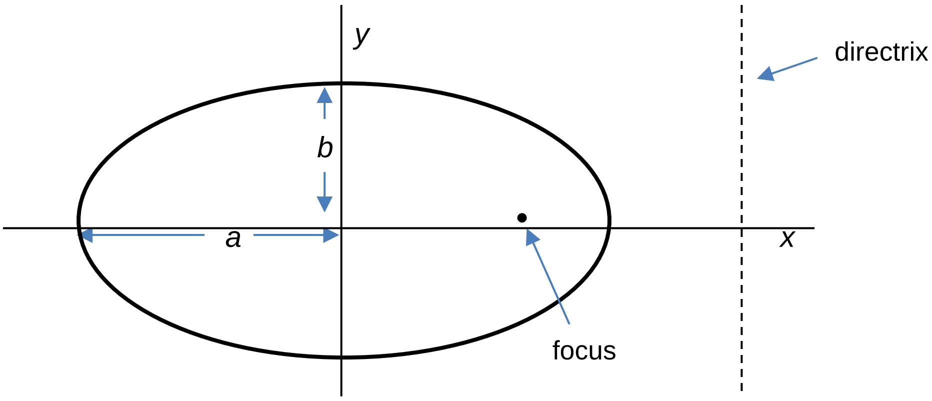
A circle around the origin has the formula $x^2 + y^2 = r^2$, where r is the radius.



An ellipse around the origin has the formula $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

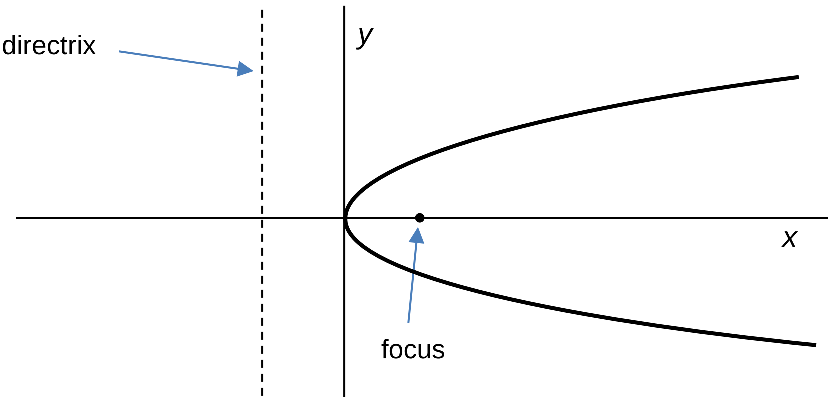
where a is the radius in the x- direction and b is the radius in the y-direction.

$e = \sqrt{1 - \frac{b^2}{a^2}}$. The directrix is the line $x = \frac{a}{e}$ and the focus is the point $x = ae$.
[By symmetry, $x = -\frac{a}{e}$ is also a directrix and $x = -ae$ is also a focus.]



We could use this same model for **a parabola** (making $e = 1$), but the focus would have to be on the end of the ellipse and both would be at infinity: there would be nothing to see near the origin.

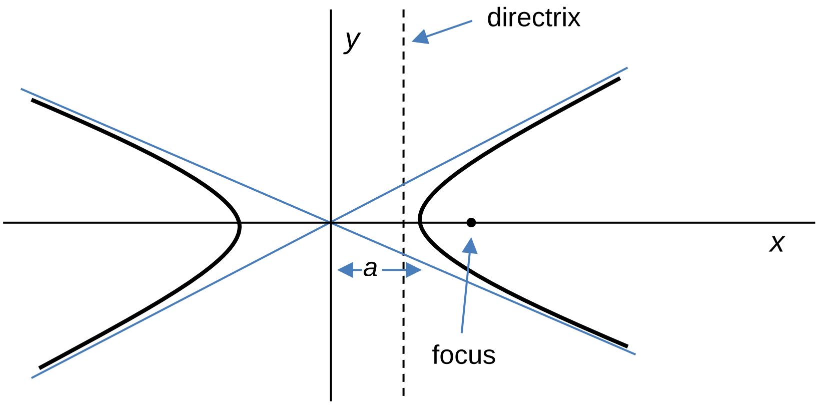
Instead, we use the function $y = x^2$ for a parabola, or, to make it more similar to the ellipse, we modify it to $y^2 = 4ax$, like this:



The focus is then at $(a, 0)$ and the directrix is $x = -a$.

A hyperbola, symmetrical about the origin has the formula $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

where a is the distance from the origin to the vertex and b is the vertical distance from the vertex to the asymptote.



$e = \sqrt{1 + \frac{b^2}{a^2}}$. The directrix is $x = \frac{a^2}{\sqrt{a^2+b^2}}$ and the focus is $x = \sqrt{a^2 + b^2}$, 0

[By symmetry, $x = -\frac{a^2}{\sqrt{a^2+b^2}}$ is also a directrix and $x = -\sqrt{a^2 + b^2}$ is also a focus.]

The gradients of the asymptotes are $\pm\frac{b}{a}$.

Practice

Q2 (a) A torch is shone straight at a wall. What shape is the lighted area?
(b) A torch whose beam has an apex angle of 50° is pointed straight at a wall,

then rotated 55°. What shape is the lighted area?

(c) Pluto's orbit brings it sometimes inside the orbit of Neptune, though more often it is further out. What shape is its orbit?

(d) A supersonic aircraft producing a shock wave in the form of a cone with the aircraft at the apex and the direction of flight along the axis flies horizontally over flat ground. What shape is the locus of all points that experience a sonic boom at exactly 10:05 a.m.?

(e) A rock is launched by catapult from the surface of the Moon so it lands 5 km away. What shape is its flight path?

Q3 For each of the four types of conic section, write from memory:

(a) the general form

(b) the expression for the eccentricity

(c) the coordinates of the focus or foci

(d) the equation(s) of the directrix or directrices

Check your answers against the text above.

Solve

Q51 Describe the locus of all points in the 3D Cartesian system (x, y, z) such that $x^2 + y^2 + z^2 = 16$

Q52 Describe the locus of all points in the 4D space-time coordinate system (x, y, z, t) such that $x^2 + y^2 + z^2 + t^2 = 4$. Describe the locus as a 3D shape evolving through time.

Q53 Stick two thumb tacks into a board covered with a sheet of paper. Make a loop of string and hook it around the thumb tacks with some slack. Pull the string out with a pencil and move the pencil around, keeping the string taut.

(a) What shape is produced?

(b) Use coordinate geometry to explain why this shape is produced.

Revise

Revision Set 1

Q61 Describe the following loci:

(a) All points in a plane equidistant from two points in the plane.

(b) All points directly north of the centre of Glasgow.

(c) All points on the Cartesian plane twice as far from the x-axis as from the point (0, 6)

(d) All points on the Cartesian plane closer to (4, 0) than to the x-axis.

(e) All points on the Cartesian plane for which the sum of the x and y coordinates is 6.

Q62 (a) A torch with apex angle of 60° is pointed straight at a wall, then turned 70°. What shape will the lit area be?

(b) Hayley's comet comes into the inner solar system every 76 years. What shape is its orbit?

(c) A space rock from outside the solar system does a fly-past of the sun. What shape is its orbit?

Q63 For each of the four types of conic section, write from memory:

(a) the general form

(b) the expression for the eccentricity

(c) the coordinates of the focus or foci

(d) the equation(s) of the directrix or directrices

Check your answers against the text above.

Answers

Q1 (a) a straight line perpendicular to the line joining the two points and half way between them
(b) a sphere of radius 3 cm
(c) the area on the Point A side of the straight line perpendicular to the AB and crossing it half way along
(d) a straight line from the origin heading on a bearing of 60°
(e) the line $y = 2x$
(f) an ellipse surrounding the point (0, 2)
(g) a parabola with a vertical line of symmetry and apex at (0, 1)
(h) a spherical shell around Point B such that Point B lies half way along its radius
(i) a square with vertices at (0, 4), (4, 0), (0, −4) and (−4, 0)
(j) the y-axis

Q2 (a) circular
(b) parabolic
(c) elliptical
(d) hyperbolic
(e) approximately hyperbolic, though, strictly, elliptical with a focus at the centre of the Moo

Q51 a spherical shell with radius 4

Q52 a spherical shell that appears with zero size at $t = -2$, expands to a radius of 2 at time $t = 0$, then shrinks back to nothing by $t = 2$.

Q53 (a) an ellipse

Q61 (a) a straight line perpendicular to the line joining the two points and half way between them
(b) a line running along the surface of the Earth from the centre of Glasgow to the North Pole
(c) an ellipse with axis of symmetry along the y-axis and vertices at (0, 4) and (0, 12)
(d) the area to the left of a hyperbola with axis of symmetry along the x-axis and vertex at (2,0), opening to the right
(e) the line $y = 6 - x$

Q62 (a) hyperbolic (b) elliptical (c) hyperbolic