M1 Maths

A6-4 Linear Programming

- graph formula inequalities on the Cartesian plane
- solve linear programming problems

Summary Learn Solve Revise Answers

Summary

A formula inequality involving two variables can be graphed on the Cartesian plane to show the area where the inequality is true.

In a linear programming problem, two variables can be adjusted to get the optimum value of an objective function. The variables are made the axes of the Cartesian plane. Constraints on the variables are represented by inequalities plotted on the plane. The area allowable by all constraints is the feasible region. The value pair in the feasible region with the optimum vale of the objective function represents the optimal values of the variables.

Learn

Formula Inequalities

The inequalities that we have looked at so far have been analogous to equations: there is one unknown and the inequality can be solved to find its possible values.

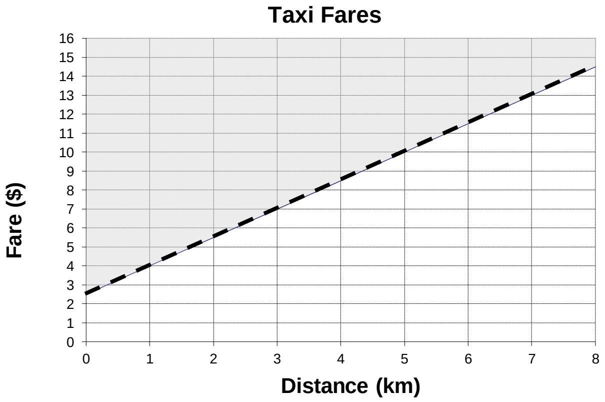
But there is another type of inequality, a formula inequlity, which is analogous to a formula with two variables. These can't be solved, but they can be graphed.

Lets day that taxi drivers have to charge a \$2.50 flag fall and \$1.50 per kilometre. The relation between fare and distance would be $f = 2.5 + 1.5d$, and we could graph this relation like so:

But suppose the drivers could charge less than that if they wanted to, though not more. The relation between fare and distance would then be $f \leq 2.5 + 1.5d$. We could graph this relation as shown below, where the shaded area represents the value pairs which are allowed.

The solid line indicates the boundary of the allowable region. The implication is that the allowable region continues off the graph to infinity below the line.

If the relation contains a ≤ or a ≥ sign, then this means that points (value pairs) on the line are allowed and we use a solid line. If the relation contains a < of > sign, then the points along the line re not allowed and we indicate this by using a dashed line. This is analogous to the solid and empty circle on the number lines. The graph of $f > 2.5 + 1.5d$ would look like this:



To graph a formula inequality, graph the formula as if the inequality sign was an = sign, using a solid line for ≤ or ≥ or a dotted line for < or >. Then decide which side of the line is allowed and shade that side. If it's difficult to decide which side, just check whether the origin (both variables zero) is on the allowable side.

Practice

Q1 Graph the following formula inequalities on separate graphs. Use suitable scales on the axes.

(a) $f < 5 + 3d$ (b) $y \geq 4x + 5$

(c) $h \leq 50c - 220$ (d) $x + y > 20$

(e) $x - 2y < 12$ (f) $2k + 3v \geq 60$

Linear Programming

Formula inequalities are used to solve optimisation problems of a type known as *linear programming*. Here is a typical linear programming problem.

> A company makes two types of wooden dining table. The *Dorchester* requires 2 hours sawing time, 6 hours assembly time and 2 hours polishing time and sells for $1200; the *Wentworth* requires 1.5 hours sawing, 3 hours assembly and 2 hours polishing and sells for $900.
>
> Constraints are imposed by the number of worker-hours available; there are only 40 worker-hours per week available for sawing, 80 for assembling and 40 for polishing. Also there is a standing order for 4 Wentworths per week, so at least 4 Wentworths need to be made. On the other hand, experience says that they will not be able to sell more than 16 Wentworths or 10 Dorchesters per week.
>
> Find the number of each type of table which should be made each week in order to maximise the money made from sales.

Linear programming problems tend to be quite involved and so are their solutions, so you might need to read this more than once.

The things we have control over are *the number of Dorchesters* and *the number of Wentworths*. We can vary these. So we call them variables. We have to pick the optimum values for them. Let's call the number of Dorchesters per week d, and the number of Wentworths per week w.

The constraints we have no control over.

What we do is we plot d against w (or vice versa – it doesn't matter).

Then we plot the constraints as inequalities in terms of d and w on the graph.

The first constraint is that there are only 40 hours sawing time available. As a Dorchester takes 2 hours sawing time and a Wentworth takes 1.5 hours, we get: $2d + 1.5w \leq 40$.

For assembly, we have: $6d + 3w \leq 80$. [Make sure you can see how we get this.]

And for polishing, we have: $2d + 2w \leq 40$.

For the standing order, we have $w \geq 4$

Because of the limits to sales, we have $w \leq 16$, $d \leq 10$

Making w the subject of the constraints gives us:

$w \le 26.67 - 1.33d, \quad w \le 26.67 - 2d, \quad w \le 20 - d, \quad w \ge 4, \quad w \le 16, \quad d \le 10$

In graphing these constraints, it can make things easier if we shade the side of the line which is not allowed. Then when all the constraints are graphed, the area which is allowed will be the area that is completely unshaded. Plotting the constraints like this gives us this graph.

The white area is the combination of Dorchesters and Wentworths which satisfy all the constraints and is thus the combinations which the company can make. For example, it can make 2 Dorchesters and 12 Wentworths or 10 Dorchesters and 6 Wentworths (as those value pairs are in the white area), but it cannot make 8 Dorchesters and 12 Wentworths (as that value pair is in the shaded area. The white area is called the feasible region.

Now we are after maximising the money made on sales. As Dorchesters sell for $1200 each and Wentworths for $900 each, the sales revenue (which we will call s), is given by $s = 1200d + 900w$. This is called the objective function as s is the quantity to be optimised.

Each point within the feasible region corresponds to a particular value for s. For example, for the point (8, 6), $s = 1200 \times 8 + 900 \times 6$, which is $15 000. What we have to do is find the point with the highest value for s.

Because s varies linearly over the feasible region, the highest value for s will be at one of the corners. (Think about that to convince yourself.). So we just find the value for s at each corner. Note that (0,16) cannot be the highest value as (4, 16) must have a higher value. Likewise with (10, 4) and (0, 4). So we only have to check the other three: (4, 15.8), (6.7, 13) and (10, 6.5).

At the first of these, $s = 4 \times 1200 + 15.8 \times 900 = 19\,020$. Similarly, for the second, $s =$ 19 740, and for the third, $s = 17\,850$

So, to maximise sales revenue, the company needs to produce 6.7 Dorchesters and 13 Wentworths per week.

Note that the values for *d* and *w* were read of the graph. If we needed them more accurately we could find them by finding the intersections of the lines from their formulae algebraically.

Note also, that the optimum production involves a fraction of a Dorchester. Should we round the number down? No. We can finish off the fraction the next week. If we need to produce $6^2/_3$ Dorchesters per week, we just produce 20 every 3 weeks.

Practice

Q2 A company makes two types of wooden dining table. The *Dorchester* requires 2 hours sawing time, 3 hours assembly time and 3 hours polishing time and sells for $1100; the *Wentworth* requires 1 hour sawing, 5 hours assembly and 1 hour polishing and sells for $800.

Constraints are imposed by the number of worker-hours available; there are only 30 worker-hours per week available for sawing, 60 for assembling and 25 for polishing. Also there is a standing order for 3 Dorchesters per week. Experience says that they will not be able to sell more than 12 Wentworths or 10 Dorchesters per week.

Find the number of each type of table which should be made each week in order to maximise the money made from sales.

Q3 Amy needs to buy some storage boxes for her stuffed toys. She has a choice of two. The smaller boxes costs $40 each, take 1.2 m^2 of floor space, and hold 1.6 m^3 of toys. The larger boxes cost $80 each, require 0.8 m^2 of floor space, and hold 2.4 m^3 of toys. She can spend a maximum of $560. Her store room has floor space for no more than 14.4 m^2 of boxes. How many of each type should she buy in order to maximize the storage volume?

Q4 An orphanage feeds the orphans just bread and tomatoes. 1 kg of bread provides 12 000 kJ of energy and 20 mg of vitamin C and costs $2.20. 1 kg of tomato provides 2000 kJ of energy and 150 mg of vitamin C and costs $6.

If the orphans are to get at least 6000 kJ and 120 mg of vitamin C daily, what combination of bread and tomatoes will make feeding them the cheapest?

Solve

Q51 Graph the formula inequality $y^2 > x^3$.

Revise

Revision Set 1

Q61 Graph the inequality $2x - 5y < 20$

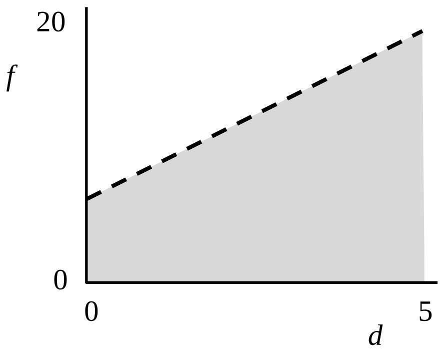
Q62 Fantasy Gardens Inc. make wooden gnomes and fairies. A gnome takes 2 hours to carve and 3 hours to paint and sells for $240. A fairy takes 4 hours to carve and 1 hour to paint and sells for $300.

Their carver works 40 hours per week and their painter works 30 hours per week. They have to make 3 gnomes and a fairy each week for a regular customer.

How many of each should they make per week to maximise sales revenue?

Answers

Q1 (a)



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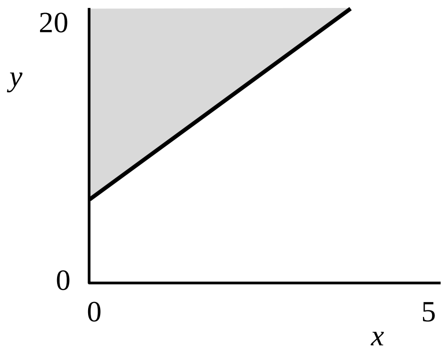

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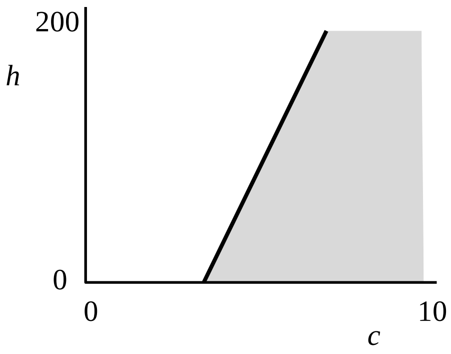

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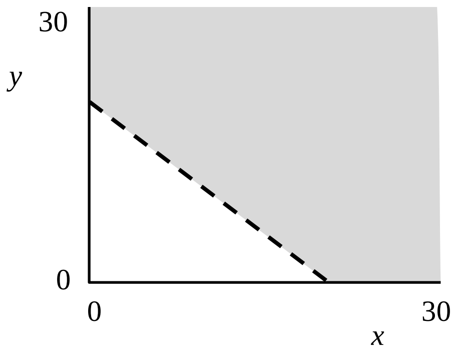

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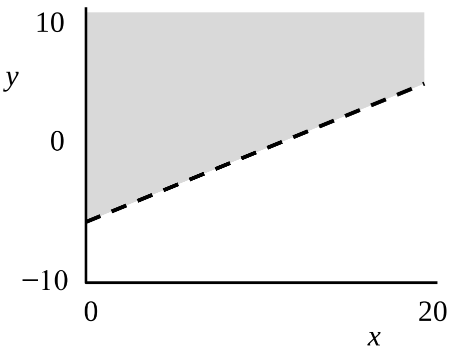

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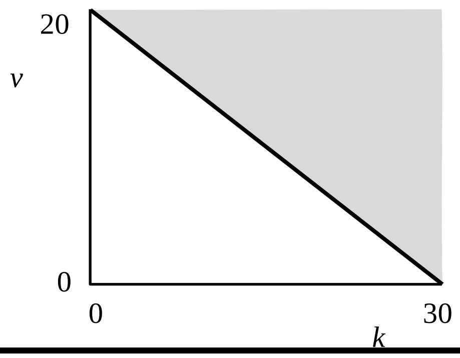
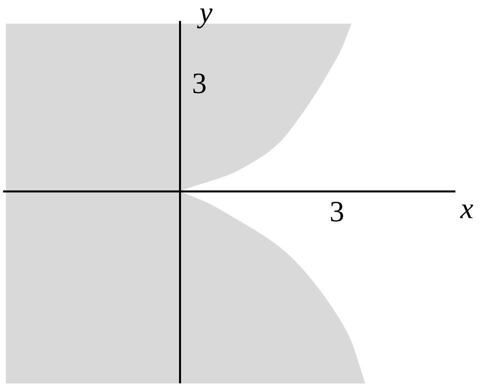
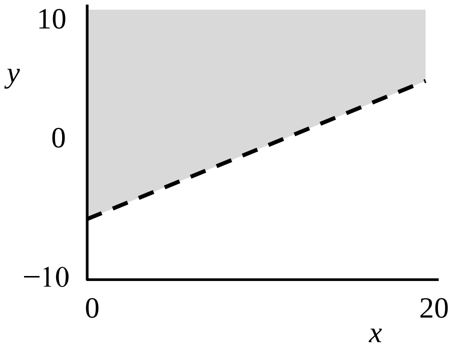


Q2 5.41 Dorchesters and 8.75 Wentworths
Q3 8 small boxes and 3 large boxes
Q4 1.042 kg of bread, 0.75 kg of tomatoes

Q51



Q61



Q62 8 gnomes and 6 fairies